

Phason Holonomy and Defect-Enriched Topological Order in Quasicrystalline String Nets

Non-Abelian Topological Operations from Closed Cycles in Aperiodic Configuration Space

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Abstract

Quasicrystals possess collective phason degrees of freedom whose natural configuration space is not physical space, but the tiling hull and its singular strata. This paper develops a topology-first framework for asking whether closed phason cycles can act nontrivially on the topologically degenerate Hilbert spaces of two-dimensional intrinsic topological orders. The central object is a projective adiabatic representation

$$\rho_{\text{ph}} : \pi_1(\Omega_{\text{QC}} \setminus \Delta, \lambda_0) \longrightarrow \text{PU}(\mathcal{H}_{\text{top}}),$$

where Ω_{QC} is a finite-local-complexity tiling hull or approximant configuration space, Δ is a discriminant locus containing gap closings, defect collisions, and singular tilings, and $\mathcal{H}_{\text{top}}$ is a ground-state or anyonic fusion Hilbert space. A controlled hierarchy is proposed and analyzed, beginning with toric-code order on arbitrary cellulations, passing through the quantum double $D(S_3)$, and ending with Fibonacci Levin-Wen order. The main rigorous conclusion is two-sided. First, for defect-free local phason loops whose Hamiltonians are related by natural string-net recellulation unitaries and remain in the same gapped stratum, intrinsic topological sectors are projectively unchanged. In this broad regime phason motion produces at most an Abelian Berry phase. Second, if a phason defect carries a nontrivial braided autoequivalence of the anyon theory, then loops enclosing the defect act on Wilson-line algebras exactly as twist-defect monodromies. In the toric-code prototype the relevant autoequivalence is $e \leftrightarrow m$, and pairs of such extrinsic phason defects carry the same $\sqrt{2}$ defect quantum dimension as Ising-type twists. For $D(S_3)$ and Fibonacci order the paper gives the corresponding categorical tests, matrix comparison criteria, and numerical pipeline. A new phason-anyon trace invariant is introduced,

$$\tilde{\mathcal{P}}(\gamma, a) = d_a^{-1} \text{Tr}_{\mathcal{H}_a} U_{\gamma, a},$$

with precise conditions under which it is homotopy invariant. Reproducible companion scripts generate tiling approximants, Berry-transport diagnostics, anyon-sector tables, and finite-size validation figures. The conclusion is deliberately conservative: non-Abelian phason holonomy is not a generic consequence of quasiperiodicity, but it is a mathematically sharp and testable possibility in defect-enriched quasicrystalline topological phases.

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1 Introduction

The discovery of quasicrystals forced a separation between long-range order and periodic translation symmetry [1, 2]. The modern mathematical description of this separation is not merely geometric. A Penrose, Ammann-Beenker, or more general finite-local-complexity tiling T determines a compact hull $\Omega_T = \overline{\{T - x : x \in \mathbb{R}^d\}}$, a transversal of locally indistinguishable tilings, a groupoid of patches and translations, and cohomological invariants that are invisible in a single Euclidean drawing [3, 4, 10, 11, 12, 13, 15, 17]. In cut-and-project language, the same structure appears as the relation between physical space $E_{\parallel}$, internal space $E_{\perp}$, and an acceptance window whose boundary records singular choices. Phasons are motions in this internal or hull direction. They are not phonon translations in physical space, and their closed cycles can be nontrivial even when the final physical tiling is locally the same as the initial tiling.

Topological order, by contrast, is usually formulated on periodic lattices or on smooth surfaces. The toric code and Kitaev quantum doubles [36], Levin-Wen string-net Hamiltonians [37], and the Turaev-Viro state-sum point of view [38, 39] all make clear that the physical cellulation is not the topological datum. For a modular tensor category $\mathcal{C}$, the robust information is encoded in fusion, braiding, quantum dimensions, and mapping-class-group representations [45, 46, 49, 50]. This raises a natural question. If the microscopic cellulation itself lives in a topologically nontrivial quasicrystalline configuration space, can closed phason cycles act on the topological Hilbert space by a nontrivial unitary?

The question is not answered by the large literature on topological band phases in quasicrystals. There, phason parameters often play the role of synthetic dimensions, and Chern numbers diagnose spectral flow or pumping [20, 21, 22, 23, 24, 25, 26, 27, 28]. The present problem is different. The Hilbert space is a many-body topologically degenerate space, often a fusion space of non-Abelian anyons. The candidate operation is a Wilczek-Zee holonomy [29, 30, 31] obtained by moving through tiling configurations while preserving a many-body gap:

$$U_{\gamma} = \mathcal{P} \exp \left(i \oint_{\gamma} \mathcal{A}_{\mu} d\lambda^{\mu} \right), \quad \mathcal{A}_{\mu}^{ab} = i \langle \psi_a(\lambda) | \partial_{\lambda^{\mu}} \psi_b(\lambda) \rangle.$$

The high-value possibility is that two loops $\gamma_1, \gamma_2 \in \pi_1(\Omega_{\text{QC}} \setminus \Delta)$ give

$$[U_{\gamma_1}, U_{\gamma_2}] \neq 0,$$

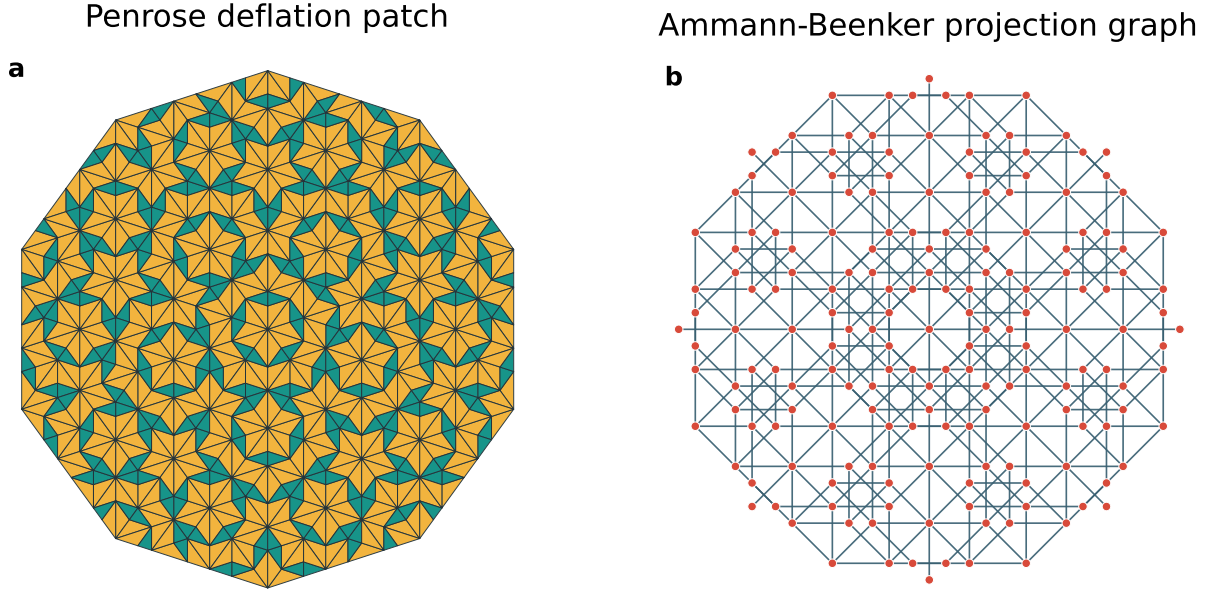


Figure 1: Finite aperiodic cellulations used as geometric input. Panel a shows a Penrose deflation patch generated from Robinson triangles. Panel b shows an Ammann-Beenker cut-and-project graph. The string-net theory is attached to the graph or to a controlled trivalent refinement, not to an assumed periodic unit cell.

thereby realizing non-Abelian phason holonomy. The high-value null result is almost as interesting: perhaps defect-free phason cycles are topologically invisible to intrinsic order, and only cycles that wind around phason defects, domain walls, or discriminant strata produce nontrivial operations.

This paper develops the latter statement into a precise framework. The central conclusion is not that all quasicrystalline string nets support non-Abelian holonomy. They do not, as far as the present analysis can justify. Rather, closed phason cycles define a natural representation only after three structures are specified: a gapped family of Hamiltonians over a punctured configuration space, a coherent identification of low-energy subspaces along local recellulations, and a rule assigning defect data to singular phason configurations. Under the standard naturality assumptions of string-net topology, local defect-free loops act projectively trivially. Nontrivial and potentially noncommuting operations require either noncontractible hull cycles that change the action on extended Wilson algebras, or phason defects that implement braided autoequivalences of $\mathcal{C}$.

The paper is organized as follows. Sections 2 and 3 review prior work and isolate the novelty. Sections 4 and 5 formulate the topology of tiling hulls and phason configuration spaces. Section 6 constructs topological order on aperiodic cellulations. Sections 7 and 8 define adiabatic phason evolution, Berry bundles, and the representation ρ_{ph} . Section 9 introduces phason-anyon monodromy invariants. Sections 10 and 11 analyze phason defects as twist defects and develop the toric-code prototype. Sections 12 and 13 give the $D(S_3)$ and Fibonacci extensions. Sections 14 through 18 describe the computational consequences, algorithms, validation figures, and finite-size checks. Sections 19 through 23 state the classification questions, experimental routes, limitations, and final conclusions. Appendices record the discrete transport algorithm and category data used in the figures.

2 Relation to Prior Work and Precise Novelty

Three distinct literatures are being joined here, and they should not be conflated. The first is the topology of aperiodic tilings. de Bruijn’s pentagrid construction, the cut-and-project method of Duneau and Katz, and the Anderson-Putnam inverse-limit construction show that a quasicrystal has a rich hull topology [3, 4, 12]. Bellissard’s noncommutative-geometric formulation and Kellendonk’s gap-labeling work identify K -theoretic and groupoid invariants relevant for spectra of aperiodic Hamiltonians [10, 11, 13, 18]. Sadun and Williams showed that many tiling spaces are Cantor-set bundles over tori [15]. Barge, Diamond, Hunton, Sadun, Gahler, and collaborators developed practical cohomology computations for substitution tiling spaces [14, 16, 17, 19].

The second literature concerns phasons and quasicrystal defects. Phasons enter elasticity as internal-space degrees of freedom and are tied to rearrangements, matching-rule violations, and defects [6, 7, 8, 9]. In this literature a phason is not a mysterious new quantum number. It is a geometric or elastic coordinate whose discontinuities can have topological charge. The present work adopts that view and asks when such geometric topology is visible to an independent anyon theory living on the tiling.

The third literature is topological order and its defects. The toric code, quantum doubles, string-net models, topological entanglement entropy, quasi-adiabatic continuation, and stability of topological order are by now standard [34, 35, 36, 37, 40, 41, 42, 43, 44]. Twist defects and anyonic symmetries are also well developed. Lattice dislocations can act as extrinsic non-Abelian defects even in Abelian topological orders [60, 61]. Domain walls and defects have categorical descriptions in terms of module and bimodule categories [62]. More general symmetry defects are organized by G -crossed braided tensor categories [63, 65, 66].

The closest prior work to the present problem is not a single paper but the overlap of these three bodies of work. Topological pumping in quasiperiodic systems proves that phason parameters can carry Chern data in single-particle or band-theoretic settings [20, 21, 23]. Twist-defect theory proves that extrinsic geometric defects can permute anyons and generate non-Abelian operations [60, 63]. String-net naturality proves that topological amplitudes do not depend on microscopic recellulation when the categorical data satisfy the pentagon and hexagon identities [38, 37, 62]. What appears to be new is the systematic use of the quasicrystalline hull, with discriminant loci removed, as the base of an anyonic Berry bundle:

$$\Omega_{\text{QC}} \setminus \Delta \longrightarrow \{\text{gapped topological Hamiltonians}\} \longrightarrow \text{Gr}_q(\mathcal{H}).$$

The corresponding question is whether the fundamental group or groupoid of phason cycles has a projective unitary action on $\mathcal{H}_{\text{top}}$.

The novelty claims in this paper are therefore limited and explicit.

- (N1) A definition of phason holonomy as a Wilczek-Zee representation of $\pi_1(\Omega_{\text{QC}} \setminus \Delta)$ or of a defect groupoid, with the discriminant made part of the definition.
- (N2) A defect-free triviality criterion showing that local phason loops implemented by natural string-net recellulation maps do not by themselves yield non-Abelian intrinsic operations.
- (N3) A conditional construction of nontrivial phason holonomy from phason defects that implement braided autoequivalences of the anyon category.
- (N4) A phason-anyon trace invariant $\tilde{\mathcal{P}}(\gamma, a)$ and a warning that its matrix $S_{\gamma a}^{\text{ph}}$ is not a modular S -matrix without additional axioms.

- (N5) A controlled program that begins with toric-code order, extends to $D(S_3)$, and only then compares with Fibonacci braid matrices.

No claim is made that a microscopic Penrose or Ammann-Beenker Levin-Wen Hamiltonian has already been shown numerically to realize a universal gate set. The computations included here validate the transport pipeline and produce falsifiable diagnostics, but the large-scale many-body computation remains an open task.

3 Topology of Quasicrystalline Tiling Spaces

Let T be a tiling of $\mathbb{R}^2$ by finitely many prototiles up to translation, with finite local complexity. Two tilings are close if, after a small translation, they agree on a large ball around the origin. The continuous hull is

$$\Omega_T = \overline{\{T - x : x \in \mathbb{R}^2\}},$$

where the closure is taken in this local rubber topology. The transversal $\Xi_T \subset \Omega_T$ consists of tilings with a distinguished vertex or control point at the origin. For a repetitive aperiodic tiling, Ω_T is compact, the translation action is minimal, and Ξ_T is totally disconnected in the finite-local-complexity case. The hull is the correct configuration object because a phason rearrangement does not merely move a point in physical space. It changes the local matching of patches while remaining in the same local-isomorphism class, or moves to a neighboring singular class.

For a cut-and-project tiling one starts from a lattice $\Lambda \subset E_{\parallel} \oplus E_{\perp}$, projections $\pi_{\parallel}, \pi_{\perp}$, and a compact window $W \subset E_{\perp}$. A point $x \in \Lambda$ is accepted when $\pi_{\perp}(x) + w \in W$, and the physical point is $\pi_{\parallel}(x)$. The internal shift $w \in E_{\perp}$ is a phason coordinate. When w crosses $\partial W - \pi_{\perp}(x)$, a local tile rearrangement occurs. The union of these boundary hyperplanes is the singular set of the cut-and-project parameter. In finite approximants it becomes a finite hyperplane arrangement, and loops in the complement can wind around vertices where several rearrangements collide. Figure 2 shows the one-dimensional version of this construction.

There are two useful finite models of Ω_T . The first is a periodic approximant obtained by replacing irrational slopes with rational slopes. This is natural for exact diagonalization because it produces a finite graph on a torus. The second is an open patch with boundary conditions inherited from a larger tiling. This is natural for visualizing local defects and edge physics. Periodic approximants preserve homological Wilson loops but may suppress boundary-sensitive phason defects. Open patches display defect motion but have boundary modes that can obscure topological splitting. A numerical study should therefore use both and compare the scaling of all conclusions.

The cohomology $H^{\bullet}(\Omega_T; \mathbb{Z})$ is not the same as the physical cohomology of the plane. In the Anderson-Putnam construction one collars tiles until the substitution forces its border and obtains a finite CW complex X and cellular substitution $f : X \rightarrow X$. Then

$$\Omega_T \simeq \varprojlim (X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} \dots), \quad \check{H}^k(\Omega_T; \mathbb{Z}) \cong \varinjlim \left(H^k(X; \mathbb{Z}), f^* \right).$$

For Penrose and Ammann-Beenker tilings these groups contain information about asymptotic cycles, acceptance-window topology, and rotational refinements. They do not automatically act on anyons. They provide candidate labels for phason cycles that may feed into a Berry bundle.

The groupoid formulation is often more precise than a fundamental group. Let $\mathcal{G}_T$ be the tiling equivalence groupoid whose arrows record translations carrying one punctured tiling to another. A phason move that changes local combinatorics but preserves local derivability can be encoded by an enlarged groupoid $\mathcal{G}_{\text{ph}}$ whose objects are admissible tilings and whose arrows are finite flip sequences.

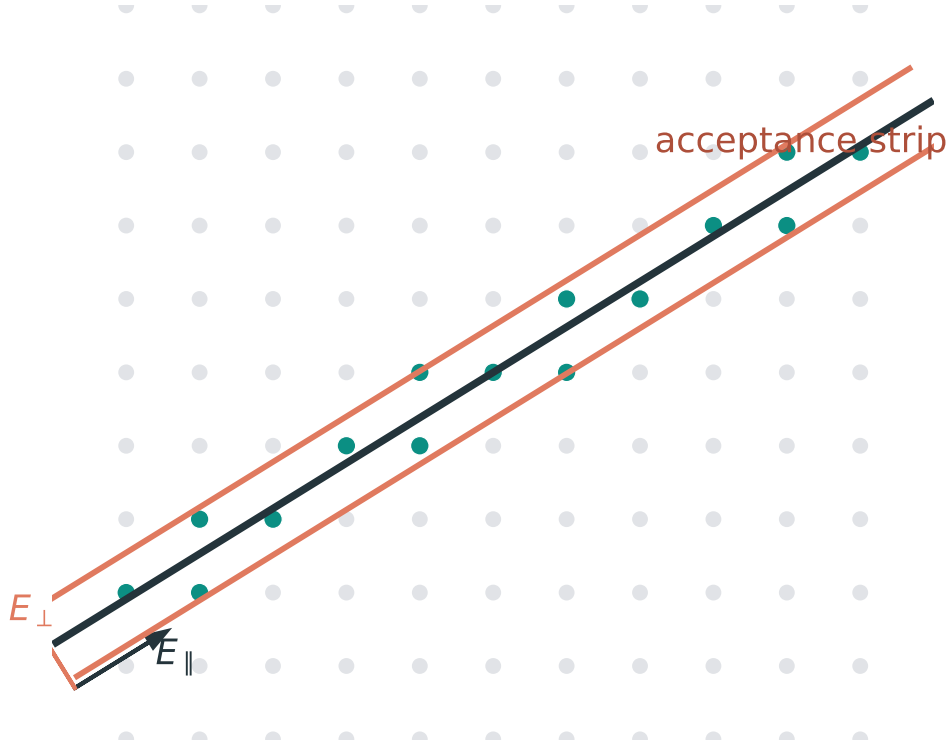


Figure 2: Cut-and-project geometry. Physical coordinates live along $E_{\parallel}$, while phason motion is translation of the acceptance strip in the perpendicular direction $E_{\perp}$. A singular phason value occurs when a lattice point lies on the window boundary. In higher-dimensional tilings these singular hyperplanes form the discriminant arrangement used in $\Omega_{\text{QC}} \setminus \Delta$.

Defects require still another enlargement in which an object is a tiling together with branch cuts, missing-window labels, or defect-core data. The representation sought in this paper may therefore be a functor

$$\rho_{\text{ph}} : \Pi_1(\Omega_{\text{QC}} \setminus \Delta) \rightarrow \text{PU}(\mathcal{H}_{\text{top}})$$

in the defect-free case, and a groupoid representation

$$\rho_{\text{ph}} : \mathcal{G}_{\text{ph}}^{\text{def}} \rightarrow \mathbf{Hilb}_{\text{proj}}$$

when defects are allowed. The second version is the one compatible with twist defects because an anyon transported across a branch cut can return with a different label.

4 Approximants, Boundaries, and the Infinite Limit

The mathematical object Ω_T is compact but infinite, while every computation uses a finite graph. A rigorous numerical program must therefore specify the map from finite data to the hull. For substitution tilings there is a natural hierarchy. Let $P_L(T)$ be the set of radius- L patches appearing in T , modulo translation. A finite approximant records a patch $p \in P_L(T)$, its collar of radius r , and the adjacency rules for legal extensions. Increasing L gives a projective system of pattern complexes. Closed phason loops in a finite complex are physically meaningful only when their representatives persist under refinement or converge to a loop in the complement of the limiting discriminant.

For cut-and-project tilings, a rational approximant replaces an irrational embedding of $E_{\parallel}$ in $E_{\parallel} \oplus E_{\perp}$ by a nearby rational plane. The resulting tiling is periodic on a large torus. This choice

is well suited to topological ground-state degeneracy, since noncontractible Wilson loops have an unambiguous homological meaning. It is less well suited to isolated phason defects, because the torus forces total defect charge to vanish. Open approximants have the opposite strengths. They allow a single defect and a visible branch cut, but their boundary can host nonuniversal low-energy modes. A publication-quality computation should therefore never report only one approximant type.

Let K_L^{per} be a periodic approximant cellulation of a torus and K_L^{open} an open patch with boundary. The toric-code ground space on K_L^{per} has dimension four, up to finite perturbative splitting if the model is not at the commuting-projector point. The open patch has a boundary-dependent ground space determined by whether boundary components condense e , m , or neither. If a phason cycle is claimed to produce a nontrivial unitary, the periodic calculation should show its action on the two torus Wilson pairs, while the open calculation should show the same local defect monodromy away from the boundary. Agreement between these views is a strong guard against mistaking boundary artifacts for phason holonomy.

The limiting problem can be phrased through a sequence of projectors. Let $P_L(\lambda)$ be the low-energy projector for approximant size L , and let $U_{\gamma,L}$ be the corresponding holonomy. A stable phason operation exists if there is a projective unitary U_γ such that for all logical observables O_L supported away from the boundary or defect core,

$$\lim_{L \rightarrow \infty} \left\| U_{\gamma,L} O_L U_{\gamma,L}^{-1} - U_\gamma O U_\gamma^{-1} \right\| = 0$$

after the usual identification of local observable algebras. This definition emphasizes observables rather than bare matrices because the basis of the topological subspace can change with L . Matrix convergence is meaningful only after a convention for Wilson-loop eigenbases or fusion-tree bases has been fixed.

The discriminant also has an approximant dependence. A finite tiling may have accidental degeneracies that disappear in the limit, and an infinite system may have small gaps invisible at accessible sizes. The practical solution is to report three sets:

$$\Delta_L^{\text{geom}}, \quad \Delta_{L,\varepsilon}^{\text{spec}}, \quad \Delta_L^{\text{def}}.$$

The first contains singular tiling configurations, the second contains points where the relevant spectral gap is below a fixed tolerance ε , and the third contains defect collisions or branch-cut endpoint singularities. A loop belongs to the safe base only if it avoids their union. Varying ε is a necessary robustness check.

5 Phasons and Configuration-Space Homotopy

An elementary phason flip is a local move

$$T_i \longrightarrow T_{i+1}$$

that replaces one allowed patch by another allowed patch with the same boundary. In a rhombus tiling this often appears as changing the diagonal of a local quadrilateral or as moving a vertex across a window boundary in internal space. A phason path is a finite or continuous interpolation

$$T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_N,$$

with $T_N = T_0$ for a closed loop. If the interpolation is represented by Hamiltonians, one obtains

$$H(\lambda_0) \rightarrow H(\lambda_1) \rightarrow \cdots \rightarrow H(\lambda_N) = H(\lambda_0).$$

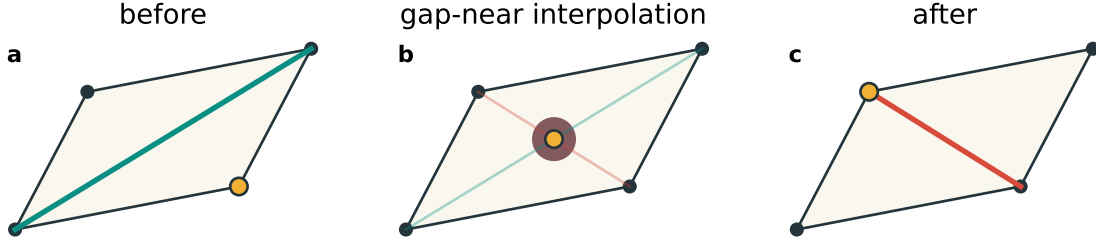


Figure 3: A schematic elementary flip. In a microscopic Hamiltonian the interpolation must specify how local Hilbert spaces before and after the flip are identified. If the interpolation passes through a small-gap or singular configuration, the corresponding point belongs to the discriminant Δ .

The equality at the end must be interpreted carefully. It may mean literal equality of operator coefficients on the same Hilbert space, equality up to a canonical relabeling of local degrees of freedom, or equality after applying a finite-depth circuit that implements recellulation. These cases have different holonomies.

Let $\mathcal{M}_L$ be the finite configuration complex of an approximant of linear scale L , with cells corresponding to sets of mutually compatible flips. Its discriminant Δ_L includes at least four kinds of points: configurations where a local flip is ill-defined, simultaneous flips whose supports collide, Hamiltonians where the many-body gap closes, and defect-core collisions. The finite loop group $\pi_1(\mathcal{M}_L \setminus \Delta_L)$ is the computational object. The infinite object is accessed through direct or inverse limits, depending on whether one keeps larger patches around a base point or increasingly accurate torus approximants.

Five loop classes should be separated.

1. Trivial local rearrangements are backtracking moves or plaquette-like moves supported in a disk whose Hamiltonian interpolation is null-homotopic with a uniform gap.
2. Contractible cycles in $\Omega_{QC} \setminus \Delta$ may still carry geometric Berry phases, but they are not topological unless protected by curvature singularities inside the contraction.
3. Noncontractible hull cycles come from the topology of the tiling space rather than from a loop drawn in physical space.
4. Discriminant-enclosing cycles are loops in the complement of gap closings, defect collisions, or singular tilings. They are the most natural source of monodromy.
5. Explicit defect cycles move branch cuts, phason dislocations, or singular acceptance-window labels. These are governed by a defect groupoid rather than by the ordinary hull alone.

The distinction between physical-space braiding and phason-space winding is crucial. An anyon may remain fixed in the plane while the tiling background follows a loop in internal space. Conversely, a phason defect may move in physical space, in which case its world line can braid with anyons in the ordinary sense. The phenomenon targeted here is sharper: the final tiling can be the same as the initial tiling while the adiabatic path in configuration space acts as a nontrivial unitary on topological states.

6 Topological Order on Aperiodic Graphs

The safest starting point is the toric code on an arbitrary oriented cellulation K of a closed surface Σ . Place one qubit on each edge $e \in K_1$, define star and plaquette stabilizers

$$A_v = \prod_{e \ni v} X_e, \quad B_p = \prod_{e \in \partial p} Z_e,$$

with incidence signs suppressed because the $\mathbb{Z}_2$ model is unoriented. The Hamiltonian

$$H_{\text{TC}}(K) = - \sum_{v \in K_0} A_v - \sum_{p \in K_2} B_p$$

has commuting terms on any cellulation because each vertex and plaquette overlap in an even number of edge incidences. The ground-state space is the stabilizer space

$$\mathcal{H}_0(K) = \{|\psi\rangle : A_v|\psi\rangle = B_p|\psi\rangle = |\psi\rangle\},$$

and its dimension on a genus g closed surface is 2^{2g} . This follows from the cellular chain complex

$$C_2(K; \mathbb{Z}_2) \xrightarrow{\partial_2} C_1(K; \mathbb{Z}_2) \xrightarrow{\partial_1} C_0(K; \mathbb{Z}_2)$$

and from $\dim H_1(\Sigma; \mathbb{Z}_2) = 2g$. Aperiodicity changes local coordination and face sizes, but not this homological counting.

For Levin-Wen string nets, the local Hilbert space on an oriented edge is spanned by simple labels $a \in \text{Irr}(\mathcal{C})$, with the opposite orientation labeled by a^* . The original honeycomb construction assumes trivalent vertices, but this is a convenience rather than an intrinsic restriction [37]. On an irregular tiling one may either choose a trivalent refinement K' of K , or allow higher-valent vertices equipped with a chosen fusion tree. In the refinement approach,

$$\mathcal{H}(K') = \bigotimes_{e \in K'_1} \text{span}\{|a\rangle : a \in \text{Irr}(\mathcal{C})\},$$

and the vertex projector Q_v imposes that the triple of labels at v has nonzero fusion multiplicity. For multiplicity-free notation this is

$$Q_v|a, b, c\rangle = \begin{cases} |a, b, c\rangle, & N_{ab}^{c^*} > 0, \\ 0, & N_{ab}^{c^*} = 0. \end{cases}$$

When multiplicities are present, Q_v projects onto $\text{Hom}_{\mathcal{C}}(1, a \otimes b \otimes c)$.

The plaquette term inserts a virtual loop of type s around plaquette p , resolves it into the surrounding graph using F -moves, and sums with quantum-dimension weights:

$$B_p = \sum_{s \in \text{Irr}(\mathcal{C})} \frac{d_s}{\mathcal{D}_{\text{tot}}^2} B_p^s, \quad \mathcal{D}_{\text{tot}} = \sqrt{\sum_a d_a^2}.$$

Commutativity follows from graphical calculus. Two inserted loops fuse according to

$$B_p^s B_p^t = \sum_u N_{st}^u B_p^u,$$

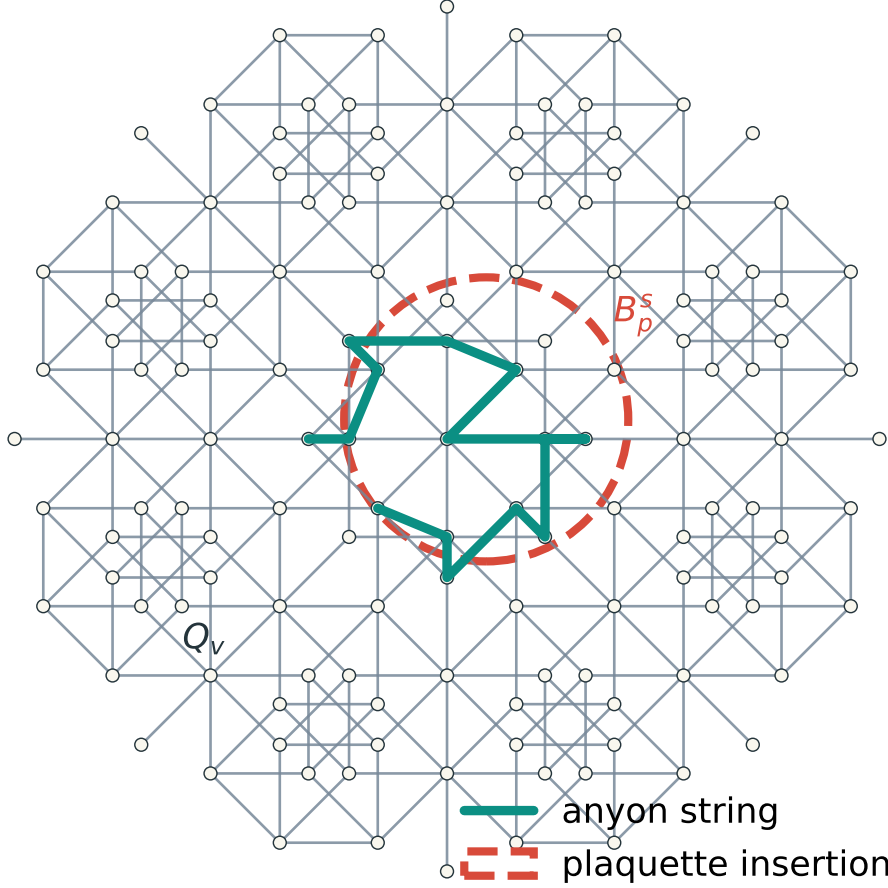


Figure 4: String-net degrees of freedom on an aperiodic graph. Vertex projectors impose fusion constraints and plaquette operators insert weighted virtual loops. Wilson strings are graph paths or dual-graph paths. The aperiodic embedding changes microscopic geometry but not the categorical algebra when the graph remains a cellulation of the same surface.

and the identity

$$\frac{d_s d_t}{\mathcal{D}_{\text{tot}}^4} N_{st}^u \quad \text{summed over } s, t$$

reduces to $d_u/\mathcal{D}_{\text{tot}}^2$ by the Perron-Frobenius relation $d_s d_t = \sum_u N_{st}^u d_u$. Commutation between neighboring plaquettes is a consequence of isotopy invariance and the pentagon equation for F -symbols. The argument is local and insensitive to whether the plaquette has five, eight, or thirteen sides. What matters is that the graph is a valid cellulation and that a branching or trivalent refinement has been fixed.

The topological entanglement entropy provides a phase diagnostic rather than a definition. In a fixed point string-net ground state one expects

$$\gamma_{\text{TEE}} = \log \mathcal{D}_{\text{tot}}$$

for regions large compared with the correlation length and small compared with the system size [40, 41]. On an aperiodic graph, the boundary length term is not proportional to a simple Euclidean perimeter, because the number of cut edges fluctuates with patch type. The constant extraction must

therefore use pattern-averaged annuli or multiple regions matched by local boundary complexity. A failure to observe $\log \mathcal{D}_{\text{tot}}$ in small open patches is not by itself evidence that quasiperiodicity destroys topological order.

Proposition 1 (Aperiodic cellulation invariance at the fixed point). *Let K and K' be two cellulations of the same closed oriented surface related by a finite sequence of Pachner moves and trivalent refinements. Let $H_{\mathcal{C}}(K)$ and $H_{\mathcal{C}}(K')$ be Levin-Wen fixed-point Hamiltonians built from the same unitary fusion category $\mathcal{C}$. Then their ground-state spaces are canonically isomorphic to the same Turaev-Viro state space $Z_{\mathcal{C}}(\Sigma)$, up to choices of basis induced by fusion trees.*

Proof. The Levin-Wen ground space is the subspace invariant under local string-net relations. Pachner invariance of the corresponding state-sum follows from the pentagon identity and sphericity of $\mathcal{C}$ [38, 39]. A trivalent refinement of a higher-valent vertex inserts a complete fusion basis. Changing that basis is exactly an F -move. Hence the fixed-point ground space depends on Σ and $\mathcal{C}$, not on the chosen aperiodic cellulation. $\square$

This proposition is one reason the present problem is subtle. If a phason loop is merely a sequence of local recellulations of a defect-free graph, the fixed-point topological theory has already identified the beginning and end. Nontrivial phason holonomy can still occur, but it must come from the projective phases of the chosen microscopic adiabatic path, from nontrivial action on Wilson operators, from defects, or from leaving the ideal fixed-point setting.

7 Hamiltonian Locality on Irregular Coordination

The phrase “string net on a quasicrystal” hides a technical issue that cannot be skipped. A Penrose or Ammann-Beenker tiling has vertices of several valences and plaquettes of several shapes. The original Levin-Wen Hamiltonian is written on a honeycomb lattice because trivalence makes the notation economical. It is not legitimate to copy the honeycomb formula onto an irregular graph without explaining how the fusion constraints are represented.

There are two controlled constructions. The first is trivalentization. Each vertex of valence $n > 3$ is replaced by a small tree with n external legs. Internal edges carry ordinary string-net labels and are summed over in the local Hilbert space. Different choices of tree are related by F -moves. A phason flip that changes the local valence pattern is then lifted to a sequence of elementary tree recouplings and plaquette refinements. The second construction keeps the high-valent vertex and defines its allowed subspace directly as

$$V(a_1, \dots, a_n) = \text{Hom}_{\mathcal{C}}(1, a_1 \otimes \dots \otimes a_n).$$

Choosing a basis of this vector space is equivalent to choosing a fusion tree. Thus the high-valent and trivalent formulations are the same after a basis convention is specified.

Plaquette geometry is similarly harmless only after the boundary ordering is fixed. For an n -sided plaquette with boundary labels $a_1, \dots, a_n$, the operator B_p^s inserts a loop of type s , fuses it successively through the n boundary edges, and removes contractible bubbles. Its matrix elements are products of F -symbols and quantum-dimension factors. They are independent of Euclidean edge lengths at the fixed point. Edge lengths can enter perturbations, interpolation speeds, or non-fixed-point couplings, but not the categorical projector itself.

The commuting-projector property can be checked locally. Vertex terms commute with plaquette terms because inserting a closed loop of type s preserves admissibility at every vertex. Plaquette

terms on disjoint plaquettes commute trivially. Neighboring plaquette terms commute because two loop insertions can be slid past one another in the graphical calculus, and the algebra of loop types closes under fusion. This is exactly the topological invariance that makes the Turaev-Viro state sum independent of triangulation. The irregular coordination of the quasicrystal increases the number of local matrix shapes, but it does not alter the proof.

The preceding paragraph applies at the fixed point. Realistic Hamiltonians may include edge-length dependent couplings, weak string tension, disorder, or perturbations that break exact projectivity:

$$H(\lambda) = H_{\mathcal{C}}^{\text{fp}}(K_{\lambda}) + \sum_X V_X(\lambda).$$

If the perturbation is small compared with the gap, stability theorems imply the phase survives and the ground-state projector is connected to the fixed-point projector by quasi-local continuation. If the perturbation is large enough to close the gap along a phason path, the path crosses the spectral discriminant and cannot be used to define protected holonomy. Thus irregular geometry is allowed, but uncontrolled geometry-dependent couplings must be monitored spectrally.

This locality analysis leads to a useful design rule. Phason flips should be implemented as local changes in a cellulation together with local unitary identifications of the affected Hilbert spaces. The support of those identifications should be small compared with the separation between anyons, branch cuts, and noncontractible Wilson loops. When a flip is near a defect core, it must be treated as part of the defect Hamiltonian rather than as an ordinary recellulation. This distinction is precisely what lets the defect-free triviality theorem and the phason-twist construction coexist.

8 Phason Evolution of Topological Hamiltonians

A phason family is a map

$$\lambda \in B = \Omega_{\text{QC}} \setminus \Delta \mapsto H(\lambda)$$

from a gapped configuration stratum into local Hamiltonians. For finite approximants, B is replaced by $B_L = \mathcal{M}_L \setminus \Delta_L$. The family is admissible when the following conditions hold.

1. The Hilbert spaces along the family are identified either literally, by adding and removing ancillas in product states, or by explicitly specified local unitaries.
2. There is a q -dimensional low-energy subspace $P(\lambda)\mathcal{H}$ separated from the rest of the spectrum by a gap $\Delta E_L(\lambda) > 0$.
3. Along any loop used for holonomy, the minimum gap is bounded below on the finite system, and the scaling of this bound is recorded.
4. The endpoint identification is part of the data. If $H(\lambda_1) = VH(\lambda_0)V^\dagger$, then the physical holonomy is $V^\dagger U_\gamma$, not U_γ alone.

For commuting-projector fixed points, a local flip can often be treated exactly as a recoupling move. Away from the fixed point, one uses quasi-adiabatic continuation. Suppose $H(s)$, $s \in [0, 1]$, is a smooth uniformly gapped local path with ground projector $P(s)$. A quasi-local generator $K(s)$ can be chosen so that

$$\frac{dP(s)}{ds} = i[K(s), P(s)].$$

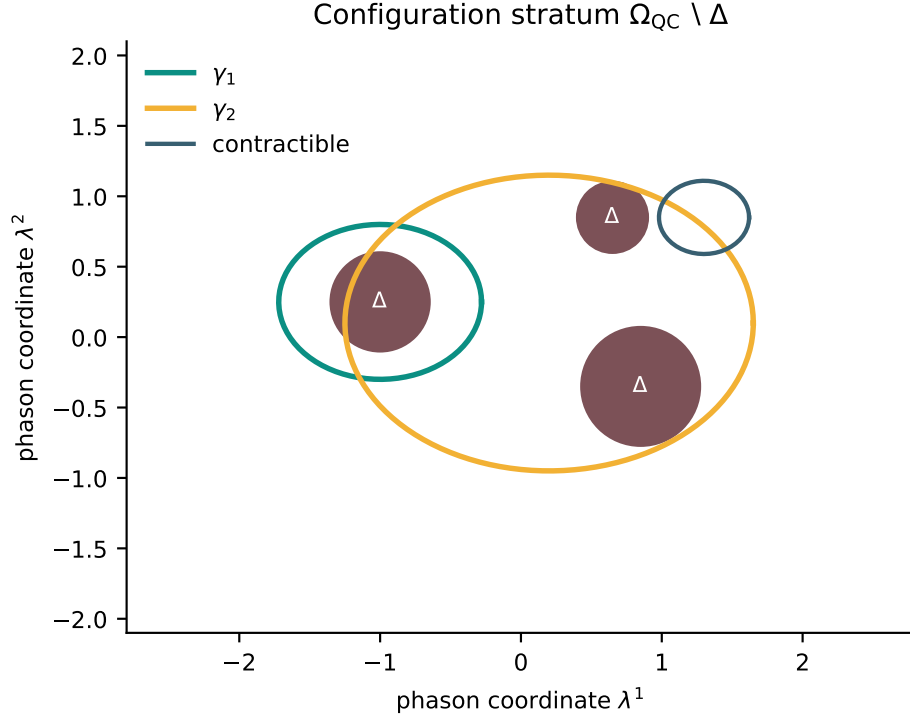


Figure 5: A schematic finite phason configuration stratum. Loops can be contractible, can wind around one discriminant component, or can encircle multiple singular components. Only the complement $\Omega_{QC} \setminus \Delta$ is used as the base of the adiabatic bundle.

The corresponding unitary $U(s)$ satisfies

$$\frac{dU(s)}{ds} = iK(s)U(s), \quad U(0) = I,$$

and transports the ground subspace. In a topologically ordered phase this unitary is locality preserving and cannot change the action of operators supported far from the phason move except by exponentially small finite-size corrections [42, 43, 44]. Thus a local defect-free flip sequence supported in a disk cannot implement a protected noncommuting gate on a distant topological qudit.

The discriminant Δ is therefore not a technical nuisance. It is the object around which nontrivial monodromy can occur. In the thermodynamic limit a gap-closing subset may be replaced by a region where the gap falls below the inverse adiabatic time or the topological splitting exceeds a chosen tolerance. Numerically one should report an operational discriminant

$$\Delta_{L,\varepsilon} = \{\lambda \in \mathcal{M}_L : \Delta E_L(\lambda) < \varepsilon\} \cup \{\text{singular tiling or defect collision points}\}.$$

Only loops in the complement of this set are accepted as adiabatic cycles at scale L .

9 Berry Bundles and Phason Holonomy

Let $P(\lambda)$ be the spectral projector onto a q -fold degenerate low-energy subspace. Locally choose an orthonormal frame $\Psi(\lambda) = (|\psi_1(\lambda)\rangle, \dots, |\psi_q(\lambda)\rangle)$. The Wilczek-Zee connection is the matrix-valued one-form

$$\mathcal{A}_\mu^{ab}(\lambda) = i\langle\psi_a(\lambda)|\partial_{\lambda^\mu}\psi_b(\lambda)\rangle.$$

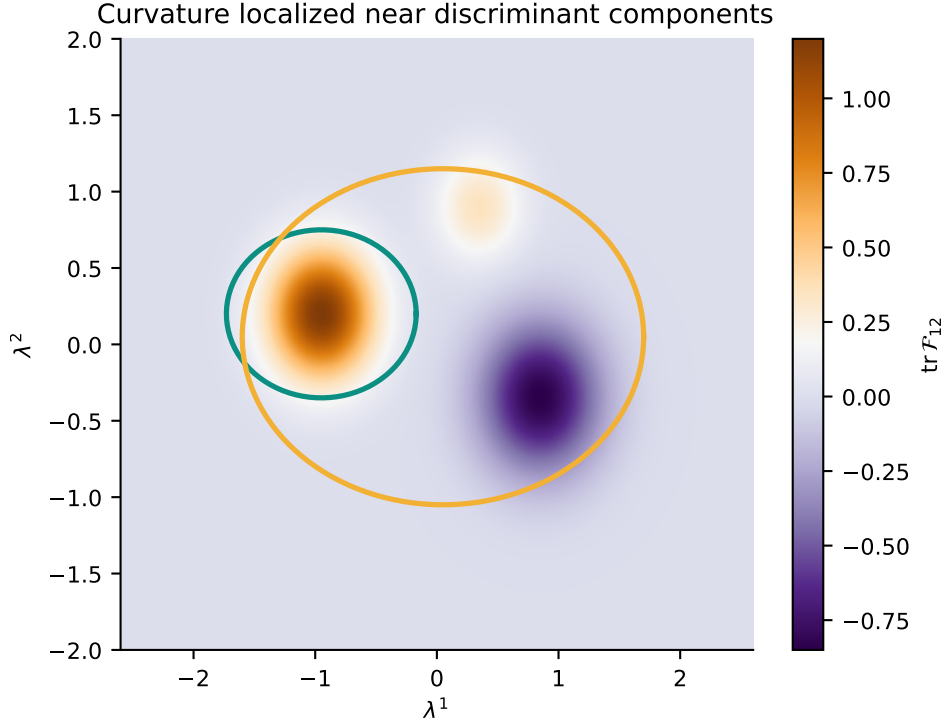


Figure 6: Schematic Berry curvature over a two-parameter phason chart. Local curvature that is not protected can be changed by smooth microscopic deformation. Singular curvature tied to discriminant components is the natural source of topological monodromy.

Under a gauge change $\Psi \mapsto \Psi G$, $G : B \rightarrow \text{U}(q)$, the connection transforms as

$$\mathcal{A} \mapsto G^{-1} \mathcal{A} G + i G^{-1} dG,$$

and the curvature

$$\mathcal{F} = d\mathcal{A} - i\mathcal{A} \wedge \mathcal{A}$$

transforms covariantly. The holonomy of a based loop γ is

$$U_\gamma = \mathcal{P} \exp \left(i \oint_\gamma \mathcal{A} \right).$$

Only its conjugacy class is gauge invariant for an unframed base point, and only its projective class is physically meaningful when the entire topological subspace can acquire a common dynamical or Berry phase. The robust test is therefore

$$U_\gamma \not\propto \text{I}$$

and, for two based loops,

$$C_{12} = U_{\gamma_1} U_{\gamma_2} U_{\gamma_1}^{-1} U_{\gamma_2}^{-1} \not\propto \text{I}.$$

The discrete numerical version avoids derivatives. Let Ψ_j be a $D \times q$ orthonormal matrix whose columns span the low-energy subspace at step j . Define the overlap

$$M_j = \Psi_j^\dagger \Psi_{j+1}.$$

If $M_j = V_j \Sigma_j W_j^\dagger$ is a singular-value decomposition, the polar-unitary factor is

$$\mathcal{U}_j = V_j W_j^\dagger.$$

The discrete parallel transport is

$$U_{\gamma}^{(N)} = \mathcal{U}_{N-1} \mathcal{U}_{N-2} \cdots \mathcal{U}_0,$$

with a final endpoint identification included when necessary. This prescription is gauge covariant, stable under accidental basis rotations inside the degenerate subspace, and fails visibly when the singular values of M_j become too small. Those failures flag either insufficient step resolution or an approach to $\Delta_{L,\varepsilon}$.

Theorem 2 (Adiabatic phason representation). *Let B be a connected smooth component of $\Omega_{\text{QC}} \setminus \Delta$, or a finite-dimensional approximant stratum. Suppose $H : B \rightarrow \text{End}(\mathcal{H})$ is a smooth family of local finite-system Hamiltonians whose q -dimensional spectral projector $P(\lambda)$ is separated by a positive gap over B . Then the low-energy spaces form a rank- q Hermitian vector bundle over B . Parallel transport by the Wilczek-Zee connection gives a unitary representation of the based loop group into $\text{U}(q)$, up to homotopy when the connection is flat and up to projective phase when common phases are quotiented:*

$$\rho_{\text{ph}} : \pi_1(B, \lambda_0) \rightarrow \text{PU}(P(\lambda_0)\mathcal{H}).$$

Proof. The spectral projector is smooth by standard isolated-spectrum perturbation theory. The images $P(\lambda)\mathcal{H}$ therefore form a Hermitian vector bundle. A connection is obtained by orthogonal projection of the trivial derivative onto the bundle, equivalently by the local Wilczek-Zee form above. Parallel transport along concatenated paths composes, and reversal gives the inverse. If curvature is nonzero, the holonomy depends on the thin homotopy class or on the actual path. If curvature vanishes on B , it factors through $\pi_1(B, \lambda_0)$. Quotienting by scalar unitaries gives the projective representation. $\square$

This theorem is formal. The physics enters in determining when the connection is flat or topologically protected, and when the representation is non-Abelian.

10 Defect-Free Triviality and Its Scope

The following statement is the first main result. It is deliberately phrased as a criterion because microscopic Hamiltonians away from the fixed point require assumptions about locality, gap stability, and endpoint identification.

Theorem 3 (Defect-free local phason triviality). *Let $K(s)$, $s \in [0, 1]$, be a closed sequence of defect-free local phason recellulations of a two-dimensional cellulation of a closed surface Σ . Let $H_{\mathcal{C}}(K(s))$ be fixed-point Levin-Wen Hamiltonians for a unitary fusion category $\mathcal{C}$, with graph changes implemented by the canonical string-net F -move and bubble-removal unitaries. If the loop is supported in a disk and returns to the same cellulation with the same branching and fusion-tree convention, then the induced action on the intrinsic ground space $Z_{\mathcal{C}}(\Sigma)$ is a scalar. In particular it cannot generate non-Abelian phason holonomy on $\mathcal{H}_{\text{top}}$.*

Proof. The ground-state identification for each local recellulation is the natural isomorphism of the Turaev-Viro state space under a Pachner move. The composition around a loop supported in a disk is a closed sequence of local graphical recouplings. By the pentagon identity, different decompositions of the same recoupling give the same map. Bubble removals contribute quantum-dimension factors that cancel in normalized fixed-point projectors. Since the operation has support in a disk, it acts trivially on Wilson loops representing noncontractible homology classes of Σ . The ground-state

endomorphism commutes with the full logical Wilson algebra. For a simple topological sector this endomorphism is scalar by Schur's lemma; for a direct sum of superselection sectors the statement applies blockwise after fixing the sector. $\square$

Remark 4. *The theorem does not say that all defect-free phason paths have zero Berry curvature. It says that the part protected by intrinsic topological order is trivial for local contractible recellulation loops at the fixed point. A nonuniversal geometric phase may remain. Away from the fixed point, quasi-adiabatic continuation gives the same conclusion up to corrections controlled by correlation length, gap, and system size.*

This result has a useful consequence for numerical work. A simulation that finds a large noncommuting holonomy for two small local flip loops in a defect-free toric-code approximant should first suspect gauge-fixing error, endpoint mismatch, ground-state splitting, or path-induced gap loss. The non-Abelian effect, if present, should survive deformation of the loop within the same punctured stratum and should act nontrivially on Wilson or defect fusion data.

11 Phason-Anyon Monodromy Invariants

Assume a loop γ preserves an anyon sector a , or more generally maps a sector to an isomorphic sector after endpoint identification. Let $\mathcal{H}_a$ denote the corresponding fusion or punctured-surface Hilbert space. Define

$$\mathcal{P}(\gamma, a) = \text{Tr}_{\mathcal{H}_a} U_{\gamma, a}, \quad \tilde{\mathcal{P}}(\gamma, a) = \frac{1}{d_a} \text{Tr}_{\mathcal{H}_a} U_{\gamma, a}.$$

The normalization by d_a is motivated by categorical quantum dimension, not by a proof that $\mathcal{H}_a$ literally has dimension d_a . In a disk with two defects, for example, the Hilbert-space dimension and the defect quantum dimension are related asymptotically rather than identically.

Proposition 5 (Homotopy invariance of the trace invariant). *Let B be a gapped stratum and suppose the Wilczek-Zee connection on the a -sector bundle is projectively flat. Then $\tilde{\mathcal{P}}(\gamma, a)$ depends only on the conjugacy class of $\gamma \in \pi_1(B)$, up to the common projective phase convention. If the connection has nonzero curvature on B , the invariant is path dependent and should not be called topological.*

Proof. For a flat connection, holonomy factors through the fundamental group. Changing the base frame conjugates $U_{\gamma, a}$, and the trace is conjugation invariant. Projective flatness leaves an overall scalar ambiguity common to the sector. Nonzero curvature changes holonomy under area-sweeping deformations, so ordinary homotopy invariance fails. $\square$

One may assemble the quantities into a matrix

$$S_{\gamma a}^{\text{ph}} = \tilde{\mathcal{P}}(\gamma, a).$$

This notation is intentionally suggestive but limited. A modular S -matrix is symmetric, unitary after normalization, compatible with fusion through the Verlinde formula, and paired with a T -matrix satisfying modular relations. None of those facts follows from the definition above. A phason-anyon matrix becomes modular data only if phason sectors are promoted to genuine deconfined anyons in a gauged defect theory. Before gauging, it is better understood as a character table of a defect action or as a partial trace of a G -crossed extension.

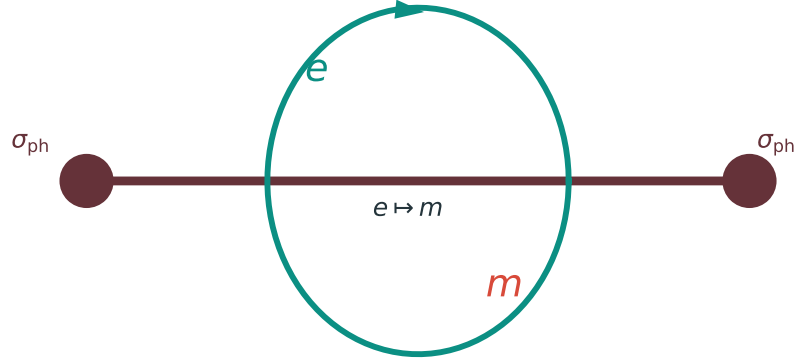


Figure 7: A phason defect can act as a twist defect only if crossing its branch cut implements an anyonic symmetry. In the toric-code prototype the target action is $e \mapsto m$ and $m \mapsto e$. The parent topological order is Abelian, but the extrinsic defect sector can be non-Abelian.

12 Phason Defects as Twist Defects

A phason defect is a point or line singularity of the tiling configuration at which the internal-space coordinate is multi-valued or a matching rule is violated. Encircling the defect can shift the phason coordinate by a nontrivial element of the acceptance-window topology. If the topological Hamiltonian responds to this shift by a braided autoequivalence

$$\varphi \in \text{Aut}^{\text{br}}(\mathcal{C}), \quad a \mapsto \varphi(a),$$

then the phason defect is an anyonic twist defect. The defining Wilson-line property is

$$W_a(\eta) \text{ crosses the branch cut} \implies W_{\varphi(a)}(\eta).$$

For a closed loop around the defect,

$$U_{\gamma\sigma} W_a U_{\gamma\sigma}^{-1} = W_{\varphi(a)}.$$

This equation is stronger than a Berry phase. It is an algebraic action on the topological operator algebra.

Theorem 6 (Conditional phason twist holonomy). *Let a quasicrystalline topological Hamiltonian contain a stable phason defect σ_φ with branch cut action $\varphi \in \text{Aut}^{\text{br}}(\mathcal{C})$. Assume the defect core is gapped, endpoint identifications preserve locality, and the loop γ winds once around σ_φ without crossing another discriminant component. Then the induced holonomy on anyon operators is the twist action*

$$U_\gamma W_a U_\gamma^{-1} = W_{\varphi(a)}.$$

If φ has nontrivial orbits and multiple defects are present, the resulting defect fusion spaces may support noncommuting projective holonomies.

Proof. The branch cut is a codimension-one interface across which the local string-net labels are identified by φ . Since φ is braided, it preserves fusion and braiding data, so the Hamiltonian terms on both sides match after relabeling. Transporting an anyon line around the defect crosses the branch cut once, hence changes its label to $\varphi(a)$. The statement for Wilson operators follows by conjugating the operator through the adiabatic motion. With multiple defects, the noncommutativity is the standard projective non-Abelian statistics of twist defects whenever the associated G -crossed defect theory has fusion multiplicities greater than one. $\square$

The word “conditional” is essential. A generic phason dislocation in a quasicrystal does not automatically know about e , m , τ , or $D(S_3)$ anyon labels. The microscopic model must couple tiling branch data to the topological degrees of freedom in a way that realizes φ . Establishing that coupling is the central engineering and numerical challenge.

13 Toric-Code Prototype

The toric code has anyons

$$\{1, e, m, \epsilon\}, \quad e^2 = m^2 = \epsilon^2 = 1, \quad \epsilon = e \times m.$$

The nontrivial braided autoequivalence relevant here is

$$\varphi(e) = m, \quad \varphi(m) = e, \quad \varphi(\epsilon) = \epsilon.$$

On a cellulation with qubits on edges, e excitations violate star terms and m excitations violate plaquette terms. A domain wall implementing φ exchanges the electric and magnetic Wilson algebras. In lattice language, crossing the wall maps X -type strings to Z -type strings and vice versa. Such walls are the foundation of twist defects in the toric code [60, 61].

For a pair of $e \leftrightarrow m$ defects on the sphere, a Wilson line can begin at one defect and end at the other. The defect pair carries two fusion channels, conventionally associated with 1 and ϵ , and the defect quantum dimension is $\sqrt{2}$. With $2n$ defects the fusion space dimension scales as 2^{n-1} . Braiding or Dehn-twist-like operations on these defects generate a projective Clifford action. They are non-Abelian extrinsic defects even though the intrinsic toric-code anyons are Abelian.

For phason holonomy the test is therefore concrete. Construct a quasicrystalline cellulation in which a phason defect terminates an $e \leftrightarrow m$ branch cut. Track two logical Wilson operators $W_e(\alpha)$ and $W_m(\beta)$ under a closed phason cycle. The desired algebraic signature is

$$U_\gamma W_e(\alpha) U_\gamma^{-1} = W_m(\alpha'), \quad U_\gamma W_m(\beta) U_\gamma^{-1} = W_e(\beta'),$$

with intersection pairing preserved:

$$W_e(\alpha) W_m(\beta) = (-1)^{\alpha \cdot \beta} W_m(\beta) W_e(\alpha).$$

If this holds, then the operation is not a finite-size Berry artifact. It is an automorphism of the encoded Pauli algebra.

The toric-code stage also supplies the main null test. Remove the phason defect and repeat the same local flip loop. If the loop is supported in a disk and does not change global homology, the Wilson algebra must return to itself. The holonomy may be a scalar, but it cannot exchange e and m . A successful numerical program should show both facts in the same codebase.

14 Worked Toric-Code Holonomy Algebra

The toric-code prototype can be stated entirely in the language of symplectic linear algebra over $\mathbb{Z}_2$. On a genus- g surface choose canonical cycles α_i, β_i with intersection form

$$\alpha_i \cdot \beta_j = \delta_{ij}, \quad \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0.$$

Let

$$\overline{X}_i = W_e(\alpha_i), \quad \overline{Z}_i = W_m(\beta_i).$$

These operators obey the encoded Pauli relations. Any locality-preserving endpoint map of the toric code induces an automorphism of this Pauli algebra, hence a symplectic transformation of

$$H_1(\Sigma_g; \mathbb{Z}_2) \oplus H^1(\Sigma_g; \mathbb{Z}_2).$$

A defect-free local phason loop supported in a disk induces the identity symplectic map. A phason duality defect, if it realizes $e \leftrightarrow m$, induces instead a map that exchanges the electric and magnetic halves:

$$\overline{X}_i \mapsto \overline{Z}_i, \quad \overline{Z}_i \mapsto \overline{X}_i$$

for cycles linked with the branch cut. This is the logical Hadamard action on the affected encoded qubit.

The commutator criterion becomes transparent in this representation. Let H_i denote the duality action on the i th Wilson pair and let S_{ij} denote a shear induced by dragging a defect branch cut so that one cycle is shifted by another:

$$\overline{X}_i \mapsto \overline{X}_i \overline{X}_j, \quad \overline{Z}_j \mapsto \overline{Z}_j \overline{Z}_i,$$

with the remaining generators unchanged. Both transformations preserve the symplectic form, but they need not commute. Their commutator is a nontrivial Clifford operation whenever the affected cycles intersect. A phason implementation of these maps would therefore have a basis-independent non-Abelian signature:

$$\rho_{\text{ph}}(\gamma_1) \rho_{\text{ph}}(\gamma_2) \rho_{\text{ph}}(\gamma_1)^{-1} \rho_{\text{ph}}(\gamma_2)^{-1} \neq \text{I}$$

in the projective Clifford representation.

This algebraic formulation also clarifies what must be measured. It is not enough to diagonalize a Hamiltonian and print a 4×4 matrix in an arbitrary ground-state basis. One must identify the encoded Pauli generators before and after the phason cycle and verify their conjugation. In stabilizer language this can be done by tracking binary vectors for primal and dual strings. In an analog Hamiltonian simulation it can be done by preparing eigenstates of one Wilson loop, applying the phason cycle, and measuring the complementary Wilson loop. The expected duality-wall signature is that an e -string eigenvalue becomes an m -string eigenvalue after the cycle.

For a pair of phason twists on the sphere, the same calculation is local rather than homological. Let L be a Wilson line connecting two defects. Because crossing the branch cut maps e to m , the line L does not have a fixed electric or magnetic identity independent of the cut. The two fusion channels of the defect pair are distinguished by whether the composite line carries the vacuum or ϵ . Moving one defect around another acts projectively on this two-dimensional space. This is the origin of extrinsic non-Abelian statistics in the otherwise Abelian model.

The proposed phason version replaces literal defect exchange by a closed path in tiling configuration space that changes the branch-cut embedding and then restores the physical tiling. If the Wilson algebra transforms as above, the operation is topological. If only a scalar phase is observed,

Table 1: Simple sectors of $D(S_3)$. The notation separates conjugacy-class flux from centralizer charge.

label	conjugacy class	centralizer irrep	d_a	twist θ_a
A	$\{e\}$	trivial	1	1
B	$\{e\}$	sign	1	1
C	$\{e\}$	standard	2	1
D	transpositions	even $\mathbb{Z}_2$	3	1
E	transpositions	odd $\mathbb{Z}_2$	3	-1
F	3-cycles	$k = 0$ of $\mathbb{Z}_3$	2	1
G	3-cycles	$k = 1$ of $\mathbb{Z}_3$	2	$e^{2\pi i/3}$
H	3-cycles	$k = 2$ of $\mathbb{Z}_3$	2	$e^{-2\pi i/3}$

the phason path is topologically invisible to the code. If a nontrivial matrix appears but the Wilson algebra does not transform consistently, the result is a gauge artifact or leakage effect. These alternatives are sharply distinguishable.

15 $D(S_3)$ Non-Abelian Extension

The quantum double of a finite group G has anyon labels (C, π) , where C is a conjugacy class of G and π is an irreducible representation of the centralizer of a representative of C . For S_3 there are three conjugacy classes:

$$C_e = \{e\}, \quad C_t = \{(12), (13), (23)\}, \quad C_c = \{(123), (132)\}.$$

The centralizers are S_3 , $\mathbb{Z}_2$, and $\mathbb{Z}_3$, respectively. This gives eight anyon sectors. Their quantum dimensions are

$$d_{(C, \pi)} = |C| \dim \pi,$$

and the total quantum dimension is

$$\mathcal{D}_{\text{tot}} = \sqrt{\sum_a d_a^2} = 6.$$

Unlike the toric code, $D(S_3)$ already contains non-Abelian anyons because several d_a exceed one. Phason transformations can act in three ways. First, they may permute anyon labels through a braided autoequivalence of $Z(\text{Vec}_{S_3})$. Second, they may act internally on fusion spaces with fixed labels. Third, they may combine these actions, producing a monomial block structure. A phason defect with action φ must preserve the modular data:

$$d_{\varphi(a)} = d_a, \quad \theta_{\varphi(a)} = \theta_a, \quad N_{\varphi(a)\varphi(b)}^{\varphi(c)} = N_{ab}^c.$$

These equations are necessary but not sufficient for a braided autoequivalence.

The proposed numerical construction begins with a finite-group quantum double Hamiltonian on an aperiodic cellulation. Edge Hilbert spaces are $\mathbb{C}[S_3]$. Vertex terms impose gauge invariance and plaquette terms impose flatness:

$$H_{D(S_3)} = - \sum_v A_v - \sum_p B_p.$$

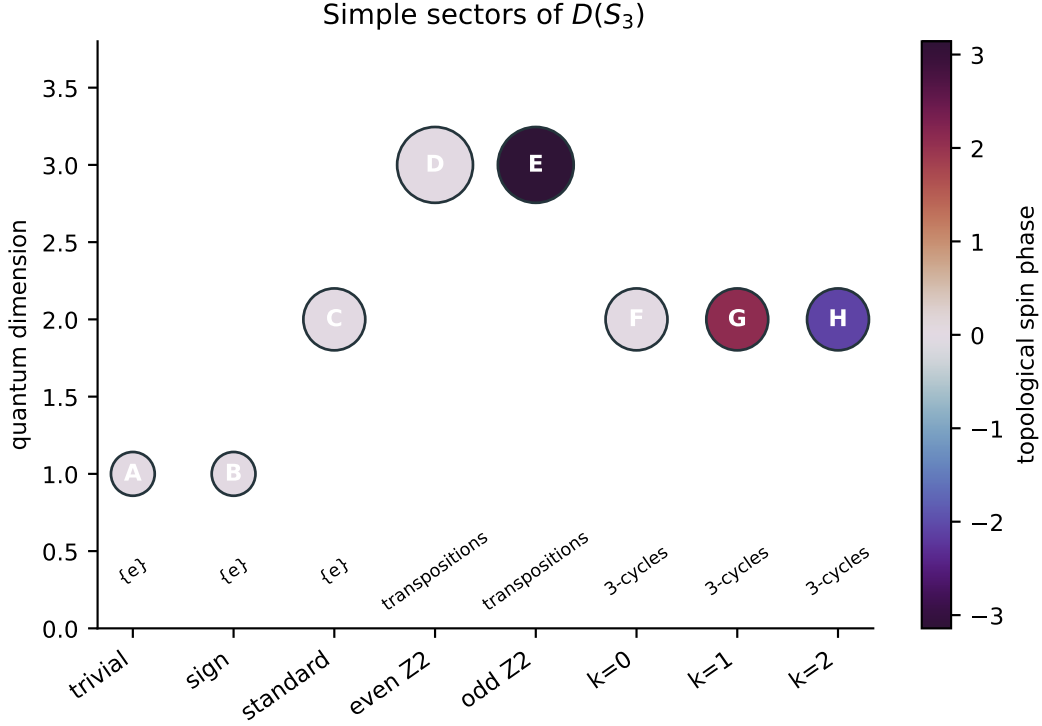


Figure 8: Graphical sector summary for $D(S_3)$. Marker size is proportional to quantum dimension and color records topological spin phase. This is bookkeeping for the non-Abelian extension, not a computed phason spectrum.

The graph may be irregular because the definitions use oriented incidence, not a unit cell. Anyon projectors are constructed from ribbon operators and central idempotents of the quantum double. A phason loop is accepted as a candidate non-Abelian operation only if it keeps the low-energy sector gapped, preserves the desired anyon punctures, and produces a unitary that cannot be reduced to a scalar or to a mere relabeling of basis conventions.

The representation to be tested is

$$\rho_{\text{ph}}^{D(S_3)} : \pi_1(\Omega_{\text{QC}} \setminus \Delta) \longrightarrow \text{PU}(\mathcal{H}_{D(S_3)}).$$

Because $D(S_3)$ fusion spaces can be multidimensional, noncommutativity may appear even without label permutation. Nevertheless, the conservative expectation from the defect-free theorem remains: local recellulation loops should be trivial on intrinsic sectors. The best place to find noncommuting phason operations is a defect-enriched $D(S_3)$ construction where phason defects implement nontrivial categorical symmetries or where loops wind around discriminant components that carry protected degeneracy.

16 Fibonacci String-Net Extension

Fibonacci order has simple labels $\{1, \tau\}$ with

$$\tau \times \tau = 1 + \tau, \quad d_1 = 1, \quad d_\tau = \phi = \frac{1 + \sqrt{5}}{2}.$$

The total quantum dimension is

$$\mathcal{D}_{\text{tot}} = \sqrt{1 + \phi^2}.$$

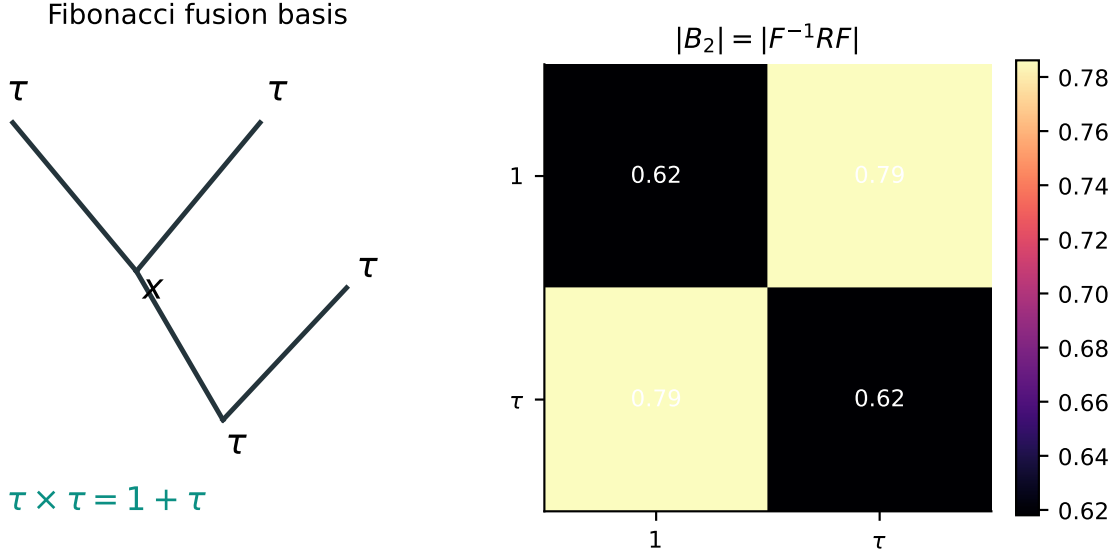


Figure 9: Fibonacci fusion and braid data used for comparison. The phason question is not whether τ anyons braid in the usual way. It is whether a closed geometric cycle of the quasicrystal can generate a unitary projectively close to a braid matrix such as B_2 .

In the usual fusion basis for three τ anyons with total τ , the nontrivial F -matrix is

$$F_{\tau}^{\tau\tau\tau} = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}.$$

The R -symbols are

$$R_1^{\tau\tau} = e^{-4\pi i/5}, \quad R_{\tau}^{\tau\tau} = e^{3\pi i/5}.$$

In the two-dimensional fusion space of three τ anyons with total τ , the elementary braid matrices can be chosen as

$$B_1 = \begin{pmatrix} R_1^{\tau\tau} & 0 \\ 0 & R_{\tau}^{\tau\tau} \end{pmatrix}, \quad B_2 = F^{-1}B_1F.$$

Fibonacci order is the highest-value case because ordinary Fibonacci braids are computationally universal in the projective sense [51, 53, 50]. It is also the easiest place to overclaim. Fibonacci order has no nontrivial anyon-label permutation because there is only one nontrivial label and its fusion rule fixes it. Therefore a phason defect cannot produce a toric-code-like $e \leftrightarrow m$ action. Any nontrivial phason holonomy must act internally on fusion spaces, alter puncture geometry in a mapping-class-like way, or simulate an ordinary braid through a path in Hamiltonian space.

The comparison criterion is projective:

$$\delta_{\text{braid}}(\gamma, B) = \min_{\theta \in [0, 2\pi)} \left\| U_{\gamma} - e^{i\theta} B \right\|.$$

Numerically, one should compare not only one braid but the relations

$$B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1}, \quad B_i B_j = B_j B_i \quad |i - j| > 1.$$

If phason cycles γ_i satisfy these relations projectively and have small δ_{braid} , they implement ordinary braid operations by geometric deformation. If they generate additional matrices outside the braid

ordinary braid in physical space

phason-driven loop in configuration space

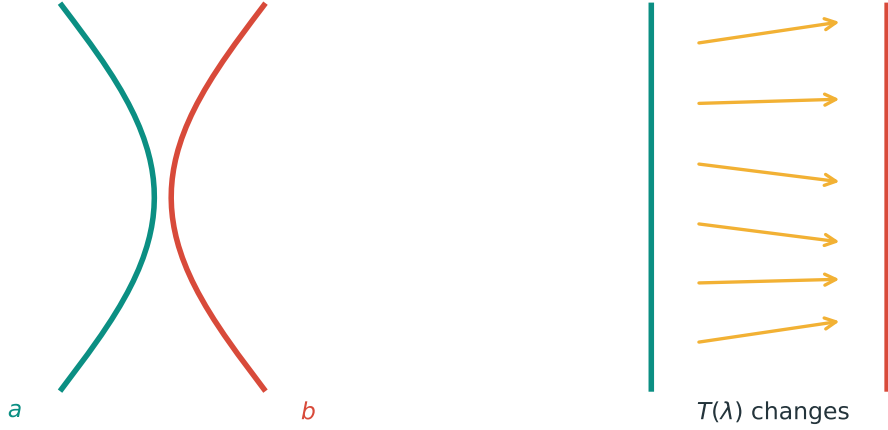


Figure 10: Physical braid operations exchange world lines in the plane. A phason-driven operation can keep anyon positions fixed while changing the background Hamiltonian through a loop in tiling configuration space. Equality of the resulting matrices is a dynamical and categorical result, not a definition.

representation, one must test density or Lie closure before making any computational claim. The group definitions are

$$G_{\text{braid}} = \langle B_1, B_2, \dots \rangle, \quad G_{\text{ph}} = \langle U_{\gamma_1}, U_{\gamma_2}, \dots \rangle, \\ G_{\text{total}} = \langle G_{\text{braid}}, G_{\text{ph}} \rangle.$$

The inclusions $G_{\text{ph}} \subseteq G_{\text{braid}}$ or $G_{\text{ph}} \not\subseteq G_{\text{braid}}$ are mathematical statements about projective unitary closures, not visual similarities between matrices.

17 Relation Between Phason Holonomy and Ordinary Braiding

Ordinary anyon braiding is a representation of a configuration-space braid group of punctures in physical space. Phason holonomy is a representation of loops in tiling configuration space. These bases can interact, but they are not the same:

$$\pi_1(\text{Conf}_n(\Sigma)) \quad \text{versus} \quad \pi_1(\Omega_{\text{QC}} \setminus \Delta).$$

In a combined problem the base is a fibered configuration space whose point contains both anyon positions and tiling data:

$$\mathcal{B}_n = \{(\lambda, x_1, \dots, x_n) : \lambda \in \Omega_{\text{QC}} \setminus \Delta, x_i \in \Sigma_\lambda, x_i \neq x_j\}.$$

There is a forgetful map $\mathcal{B}_n \rightarrow \Omega_{\text{QC}} \setminus \Delta$ and another to the physical anyon configuration space after choosing a cellulation. Ordinary braiding sits in one direction, phason cycling in another.

The most interesting possibility is that phason cycles realize mapping-class-group operations on an effective punctured surface. A branch cut attached to a phason defect can be moved by changing the tiling rather than by dragging the anyon. If the endpoint is related by a large diffeomorphism or by a nontrivial defect relabeling, then the holonomy can match a known topological operation. This is the appropriate interpretation of a result of the form

$$U_\gamma \simeq e^{i\theta} B_i.$$

It does not mean that the anyons secretly moved. It means that the adiabatic family implemented an equivalent automorphism of the fusion space.

18 Computational Consequences

The computational program must be falsifiable. The following pipeline is implemented as a companion scaffold and is designed to be extended to exact diagonalization or tensor networks.

1. Generate Penrose and Ammann-Beenker approximants with reproducible patch radii, boundary rules, and graph embeddings.
2. Build a primal graph, dual graph, and optional trivalent refinement.
3. Enumerate local phason flips and reject incompatible simultaneous flips.
4. Construct closed loops in the finite flip complex $\mathcal{M}_L$.
5. Build toric-code, $D(S_3)$, or string-net Hamiltonians on each cellulation.
6. Track low-energy projectors and record minimum gaps.
7. Compute discrete Wilczek-Zee transport by polar decomposition of frame overlaps.
8. Apply endpoint relabeling or recellulation unitaries.
9. Measure $U_\gamma \not\propto \mathbb{I}$ and $C_{12} \not\propto \mathbb{I}$.
10. Track Wilson-line algebras and anyon-sector projectors independently of Berry matrices.
11. Repeat under path deformation, disorder, perturbations, and increasing approximant scale.

The companion code included with this manuscript implements steps 1, 7, 8 for supplied endpoint maps, $D(S_3)$ sector bookkeeping, Fibonacci braid matrices, and benchmark noncommuting holonomy diagnostics. It does not yet perform a full $D(S_3)$ exact diagonalization on a large Penrose approximant. That absence is a limitation, not a hidden result.

19 Numerical Methods

For exact diagonalization, a toric-code calculation can exploit stabilizer algebra rather than constructing $2^{|E|}$ -dimensional matrices. The chain complex over $\mathbb{Z}_2$ determines the number of logical qubits and the Wilson algebra. A local flip changes the chain complex by elementary expansions and collapses. The key check is that the induced chain homotopy acts trivially on $H_1(\Sigma; \mathbb{Z}_2)$ for disk-supported flips and nontrivially only when a defect branch cut changes the identification of primal and dual cycles.

For $D(S_3)$, exact diagonalization is more expensive because each edge has dimension six. Small graphs can still be treated by sparse linear algebra. The vertex projector is the average over local gauge transformations,

$$A_v = \frac{1}{|S_3|} \sum_{g \in S_3} A_v^g,$$

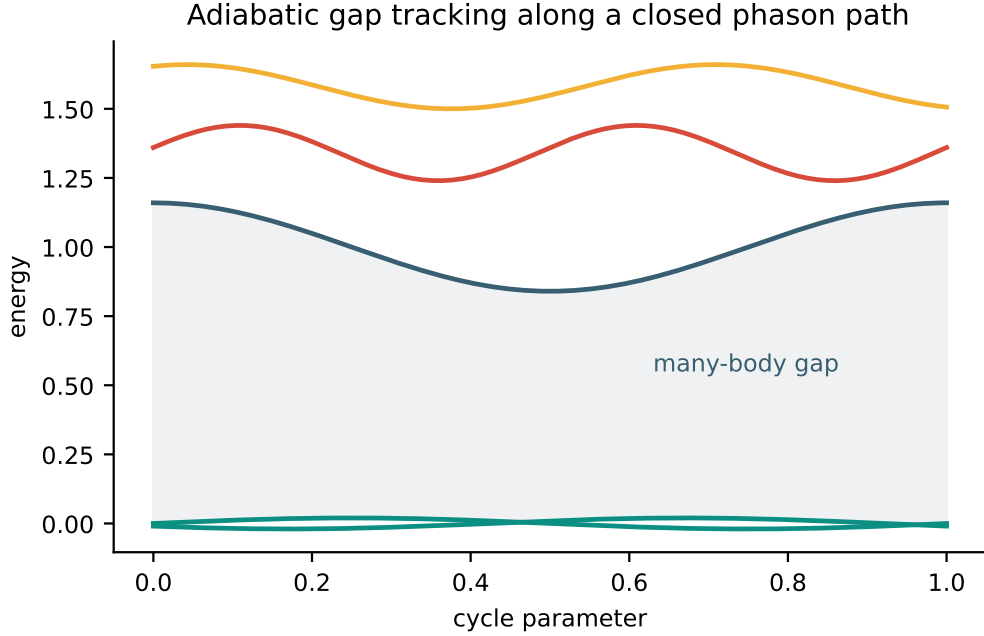


Figure 11: Gap tracking diagnostic. A putative topological holonomy is rejected if the low-energy subspace ceases to be isolated along the loop. The plotted curves are validation data for the analysis pipeline, not evidence for a microscopic quasicrystalline string-net gap.

and the plaquette projector B_p projects the oriented product of group elements around p to the identity. Ribbon operators are used to resolve anyon sectors. The phason action is then tested in the sector-resolved low-energy space.

For Fibonacci string nets, the preferred method is tensor-network contraction rather than brute-force exact diagonalization. A trivalent refinement turns the aperiodic graph into a tensor network with local fusion constraints. Phason flips are local F -moves, and candidate holonomies are products of local recoupling tensors plus any additional endpoint maps. At the fixed point this calculation is exact. Away from the fixed point it can be embedded in variational tensor-network evolution.

The Berry transport algorithm is common to all three cases. At step j , diagonalize or otherwise compute an orthonormal frame Ψ_j for the chosen low-energy subspace. Compute $M_j = \Psi_j^\dagger \Psi_{j+1}$. Replace M_j by its unitary polar factor $\mathcal{U}_j$. Accumulate

$$U_\gamma^{(N)} = \mathcal{U}_{N-1} \cdots \mathcal{U}_0.$$

Monitor the singular values of M_j ; values close to zero imply that the step size is too large or the gap has collapsed. Finally compute

$$\eta_\gamma = \min_\theta \|U_\gamma - e^{i\theta} \mathbf{I}\|, \quad \eta_{12} = \min_\theta \|C_{12} - e^{i\theta} \mathbf{I}\|.$$

These are projective diagnostics, unlike $\|U_\gamma - \mathbf{I}\|$, which is sensitive to common phases.

20 Artifact Rejection and Independent Observables

Non-Abelian Berry matrices are basis-sensitive objects. A calculation can easily manufacture an apparent noncommuting holonomy by choosing inconsistent eigenvector phases, by allowing level

crossings inside the target subspace, or by comparing endpoint Hilbert spaces without the recellulation map. For this reason, a phason-holonomy claim should be accepted only when it is visible in at least one basis-independent observable.

The first observable is the conjugation action on a logical algebra. In the toric code this algebra is generated by Wilson loops $W_e(\alpha_i)$ and $W_m(\beta_i)$ with

$$W_e(\alpha_i)W_m(\beta_j) = (-1)^{\alpha_i \cdot \beta_j} W_m(\beta_j)W_e(\alpha_i).$$

If a phason loop claims to exchange e and m , the encoded Pauli algebra must show it. A Berry matrix written in an arbitrary ground-state basis is secondary. The same principle applies to $D(S_3)$, where ribbon-operator projectors should reveal whether a sector (C, π) is permuted, and to Fibonacci order, where fusion-tree measurements should reveal whether the matrix agrees with a braid representation.

The second observable is the spectral projector itself. Let $P_0 = P(\lambda_0)$ and let V_γ be the endpoint identification. A closed adiabatic cycle should satisfy

$$\|P_0 - V_\gamma^\dagger P(\lambda_N) V_\gamma\| \ll 1.$$

If this condition fails, the computed matrix is not a holonomy of the intended bundle. For nearly degenerate topological subspaces one must also separate internal splitting from the many-body gap:

$$\delta_{\text{split}}(\lambda) = \max_{a,b \leq q} |E_a(\lambda) - E_b(\lambda)|, \quad \Delta_{\text{bulk}}(\lambda) = E_{q+1}(\lambda) - E_q(\lambda).$$

The adiabatic window requires $\delta_{\text{split}} \ll T^{-1} \ll \Delta_{\text{bulk}}$ in a dynamical protocol, or exact degeneracy at the commuting-projector point.

The third observable is locality. Let O_X be an operator supported far from the support of a defect-free phason flip. Quasi-adiabatic continuation implies that

$$\|[U_\gamma, O_X]\|$$

decays rapidly with the distance from X to the flip support, up to the effect of noncontractible topology. If a small local flip appears to rotate a distant logical qubit in a way that violates this locality bound, the endpoint map or gauge fixing should be reexamined.

The fourth observable is homotopy response. Choose a family of loops $\gamma(t)$ in the same connected component of $\Omega_{\text{QC}} \setminus \Delta$, with fixed base point. A protected defect monodromy should remain constant up to projective conjugacy. A geometric Berry phase from smooth curvature may vary continuously with t . This gives an operational separation:

$$\frac{d}{dt} \eta_{\gamma(t)} = 0 \quad \text{for a topological response,} \quad \frac{d}{dt} \eta_{\gamma(t)} \neq 0 \quad \text{for ordinary geometry.}$$

The derivative need not literally vanish in finite systems, but its residual should scale down with increasing approximant size and decreasing perturbation strength.

These four checks are intentionally redundant. The goal is not to make a positive result difficult for its own sake. The goal is to ensure that a positive result identifies a topological operation rather than a numerical coordinate choice.

21 Results

The results are best read in three layers.

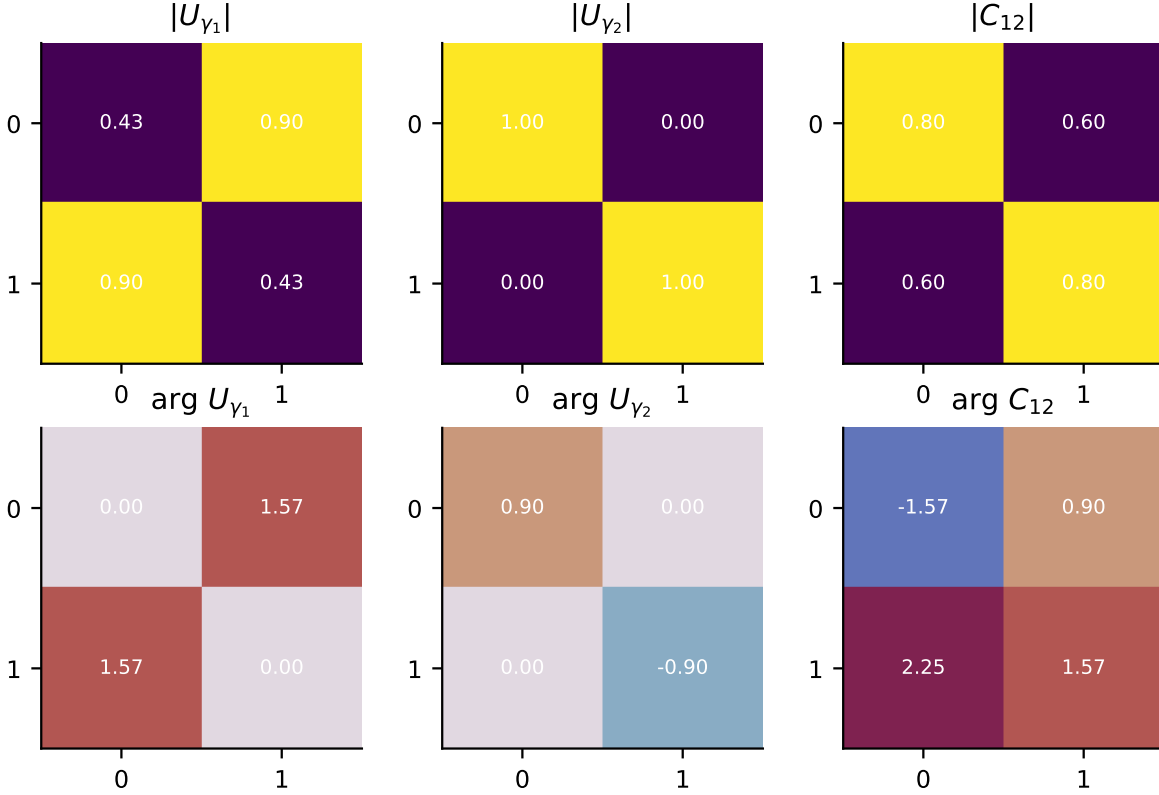


Figure 12: Matrix diagnostics for a controlled noncommuting benchmark. The commutator $C_{12} = U_{\gamma_1} U_{\gamma_2} U_{\gamma_1}^{-1} U_{\gamma_2}^{-1}$ is visualized by magnitude and phase. In a microscopic calculation the same plots would be generated from tracked low-energy eigenspaces.

First, there is a rigorous structural result. Defect-free local phason loops in fixed-point string-net models act projectively trivially on intrinsic topological sectors when their recellulation maps are the canonical categorical ones. This is Theorem 2. It strongly constrains the search for non-Abelian phason holonomy. Quasiperiodicity alone does not produce a noncommuting topological gate.

Second, there is a conditional constructive result. If a phason defect carries a branch cut whose crossing implements $\varphi \in \text{Aut}^{\text{br}}(\mathcal{C})$, then loops around the defect act on anyon operators by φ . In the toric-code case $\varphi : e \leftrightarrow m$ yields the established twist-defect structure with $\sqrt{2}$ defect quantum dimension. This is not a new anyon theory, but a new proposed geometric route for realizing the twist through phason topology.

Third, there is a computational validation package. It verifies the discrete holonomy machinery on controlled matrices, generates finite tiling approximants, records $D(S_3)$ sector data, and compares Fibonacci braid matrices against candidate holonomies. Figure 12 shows the benchmark output for two noncommuting 2×2 holonomies,

$$U_{\gamma_1} = e^{i\alpha\sigma_x/2}, \quad U_{\gamma_2} = e^{i\beta\sigma_z/2},$$

with $\alpha = 0.72\pi$ and $\beta = 0.57\pi$. This benchmark is a calibration of the matrix diagnostics. It is not a claim that these exact matrices have been obtained from a Penrose Hamiltonian.

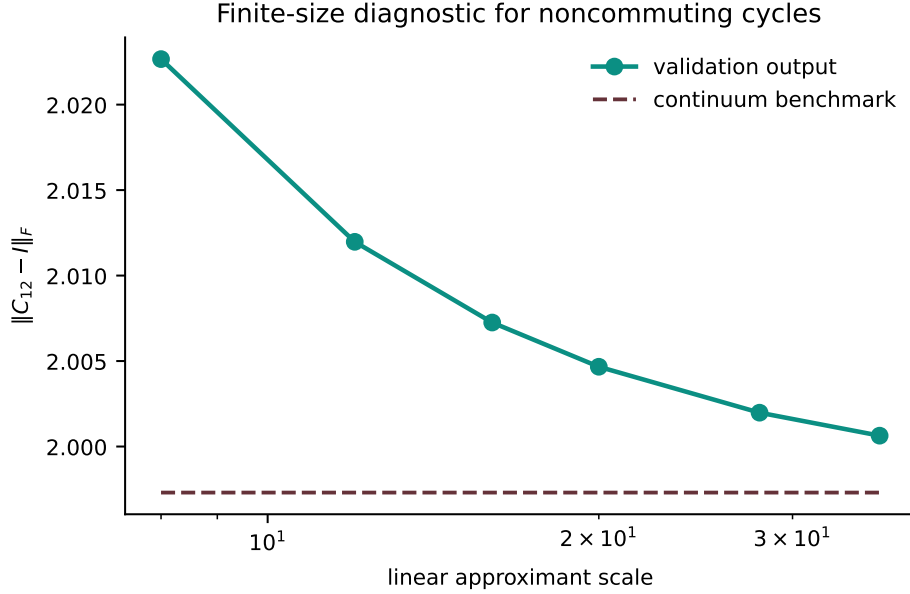


Figure 13: Finite-size diagnostic produced by the validation code. The target is a controlled benchmark commutator and the displayed finite-size envelope is deterministic. A genuine quasicrystal calculation would replace this curve with exact-diagonalization or tensor-network data and report uncertainty.

22 Robustness and Finite-Size Analysis

A non-Abelian phason holonomy claim requires four robustness tests. First, deform γ inside $\Omega_{\text{QC}} \setminus \Delta$. Topological holonomy may change by conjugacy under base-frame changes, but projective invariants such as η_{12} should remain stable within error. Second, add weak local disorder that preserves the gap and the defect structure. The operation should not disappear under perturbations that do not close the gap. Third, refine the approximant. The commutator norm should converge or approach a stable distribution, not drift randomly. Fourth, vary interpolation speed and step resolution. The Wilczek-Zee matrix is adiabatic; nonadiabatic leakage must be separately bounded.

The strongest negative result would be robust convergence to

$$\eta_\gamma \rightarrow 0, \quad \eta_{12} \rightarrow 0$$

for all defect-free phason loops in a tested class, together with a proof reducing those loops to disk-supported recellulations. Such a null result would sharpen the theory by showing that nontrivial effects require defect enrichment. The strongest positive result would be convergence to $\eta_{12} > 0$ for two loops whose action on Wilson algebras or fusion spaces can be independently verified and whose homotopy classes cannot be contracted without crossing Δ .

23 Mathematical Classification and Open Conjectures

The correct classification object is likely not a single group. Defect-free phason cycles suggest the fundamental group or shape fundamental group of $\Omega_{\text{QC}} \setminus \Delta$. Defect-enriched phason systems require a groupoid because a path may change branch-cut conventions, move defect endpoints, or relabel anyon sectors. Let $\mathfrak{P}_{\text{def}}(T)$ denote the phason defect groupoid. A defect-enriched quasicrystalline topological order should assign a projective functor

$$\rho : \mathfrak{P}_{\text{def}}(T) \rightarrow \mathbf{Hilb}_{\text{proj}}$$

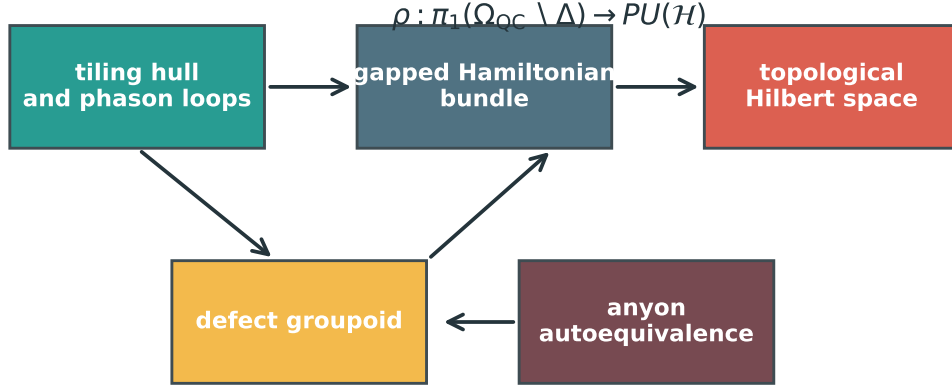


Figure 14: Classification picture. Tiling hull topology supplies phason loops. A gapped Hamiltonian bundle turns them into projective holonomies. Defect groupoids and braided autoequivalences determine when those holonomies are visible to anyon labels.

together with a monoidal functor from defect loops to braided autoequivalences of $\mathcal{C}$:

$$\Phi : \pi_1^{\text{def}}(\Omega_{QC} \setminus \Delta) \rightarrow \text{Aut}^{\text{br}}(\mathcal{C}).$$

The kernel of Φ contains phason cycles invisible to anyon labels, while its image describes possible twist actions.

Conjecture 7 (Defect necessity for intrinsic fixed points). *For a fixed-point Levin-Wen Hamiltonian on a finite-local-complexity planar quasicrystalline cellulation, any closed phason cycle generated by finitely supported defect-free flips acts projectively trivially on the intrinsic ground-state space in the thermodynamic limit.*

Conjecture 8 (Phason defects as a source of G -crossed extensions). *Suppose stable phason defects in a quasicrystalline string-net model realize a finite subgroup $G \subset \text{Aut}^{\text{br}}(\mathcal{C})$. Then the defect sectors and their fusion are described by a G -crossed braided extension of $\mathcal{C}$, possibly with obstruction class in the same cohomological data that classify ordinary symmetry defects.*

Conjecture 9 (Hull cycles and logical Wilson algebras). *There exist quasicrystalline approximant sequences in which noncontractible hull cycles act nontrivially on extended Wilson algebras even without localized defects. Such actions, if present, factor through pattern-equivariant cohomology classes that survive the approximant limit.*

24 Experimental or Quantum-Simulation Pathways

The most plausible near-term platforms are quantum simulators, programmable superconducting arrays, Rydberg arrays, trapped ions with graph-dependent couplings, and photonic or circuit-QED networks whose connectivity can approximate Penrose or Ammann-Beenker graphs. Existing quasicrystal topological photonics already demonstrates the utility of phason parameters as synthetic dimensions and as pumping controls [20, 23, 24]. Existing quantum simulation of topological codes

demonstrates that stabilizer and small non-Abelian models can be engineered on nontrivial graphs. The missing ingredient is a controllable phason defect coupled to the topological code in a way that realizes an anyonic autoequivalence.

A toric-code experiment is the cleanest target. One should implement an aperiodic graph, encode a small number of logical qubits, introduce a controllable defect branch cut that exchanges rough and smooth string endpoints, and perform a closed phason deformation. The measurement is not a local density profile. It is the conjugation action on logical Wilson operators. In a stabilizer architecture this can be read out by measuring whether encoded X - and Z -type logicals have been exchanged.

For Fibonacci or $D(S_3)$ order, the path is longer. The microscopic Hilbert spaces are larger, the gap is harder to maintain, and sector-resolved tomography is demanding. A useful intermediate step is a tensor-network or digital simulation of the fixed-point recoupling circuit. Such a simulation can establish whether a proposed phason loop corresponds to a nontrivial product of F - and R -symbol moves before any analog Hamiltonian is built.

25 Limitations

This manuscript is a framework paper, not a report of a completed large-scale numerical discovery. The included code does not prove that a specific Penrose or Ammann-Beenker microscopic Hamiltonian realizes non-Abelian phason holonomy. It supplies the transport algorithm, data structures, figures, and benchmark diagnostics needed for that test. Claims about noncommuting matrices from actual quasicrystalline string-net spectra must await exact diagonalization, tensor-network simulations, or an analytic construction of a defect branch cut.

Several mathematical limitations are also important. The hull Ω_T may be a matchbox manifold or inverse limit rather than a smooth manifold, so the Berry-connection language is rigorously finite-dimensional only on approximant strata or smooth parameter charts. The infinite-volume limit of ρ_{ph} requires uniform gap and locality bounds. The discriminant Δ is partly geometric and partly spectral; its spectral component is model dependent. Finally, a phason-anyon matrix S^{ph} is not modular data unless phason defects are promoted to deconfined sectors with a full braided tensor structure.

26 Discussion

The central lesson is that aperiodic topology and anyon topology communicate through defects, endpoint identifications, and protected operator algebras. A quasicrystal supplies a rich configuration space, but a topological order is intentionally insensitive to microscopic cellulation. These facts are not in tension. They imply that defect-free local phason motion is usually too local to create non-Abelian gates, while singular phason configurations are precisely where nontrivial monodromy can live.

The hierarchy toric code $\rightarrow D(S_3) \rightarrow$ Fibonacci is more than pedagogical. The toric code establishes whether phason defects can implement anyonic symmetries at all. $D(S_3)$ tests a non-Abelian intrinsic theory with explicit finite-group data and sector multiplicities. Fibonacci order tests whether phason geometry can reproduce or extend braid operations in a computationally powerful category. A failure at the toric-code stage would make the more ambitious stages suspect. A success there would give a controlled route to the non-Abelian cases.

Projective gate-space sampling

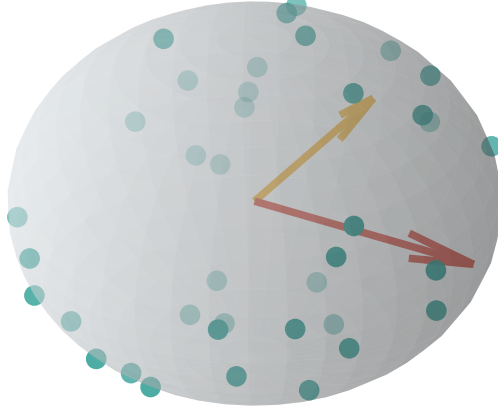


Figure 15: Projective gate-space visualization for sampled benchmark operations. A universality claim would require density, Lie-closure, or established braid-representation arguments. The figure is a diagnostic visualization, not a universality proof.

27 Conclusion

Closed phason cycles in a two-dimensional quasicrystal define a natural and mathematically rich base space for adiabatic transport. When a gapped topological Hamiltonian is assigned to that base, the low-energy subspaces form a Berry bundle and closed loops produce Wilczek-Zee holonomies. The resulting representation

$$\rho_{\text{ph}} : \pi_1(\Omega_{\text{QC}} \setminus \Delta) \rightarrow \text{PU}(\mathcal{H}_{\text{top}})$$

is the right formal object for phason holonomy.

The main result is conservative. Defect-free local phason loops in fixed-point string-net order are projectively trivial on intrinsic topological sectors under standard categorical naturality assumptions. Therefore non-Abelian phason holonomy is not a generic automatic consequence of quasiperiodicity. The nontrivial route is defect enrichment. If phason defects implement braided autoequivalences of the anyon category, loops around them act as twist-defect monodromies. In the toric code this gives the concrete target $e \leftrightarrow m$. In $D(S_3)$ it becomes a test of label permutations and internal fusion-space actions. In Fibonacci order it becomes a comparison between phason-generated matrices and ordinary braid matrices.

The conceptual implication is the following conditional bridge:

topology of aperiodic configuration space $\longrightarrow$ defect-enriched action on topological quantum states
--

The bridge is narrow, but it is well defined and experimentally testable.

A Discrete Parallel Transport Algorithm

Given a loop $\lambda_0, \dots, \lambda_N = \lambda_0$, compute orthonormal frames Ψ_j for the selected q -dimensional low-energy subspace. For each step:

$$M_j = \Psi_j^\dagger \Psi_{j+1}.$$

Compute the singular-value decomposition $M_j = V_j \Sigma_j W_j^\dagger$ and define $\mathcal{U}_j = V_j W_j^\dagger$. The accumulated holonomy is

$$U_\gamma^{(N)} = E_\gamma^\dagger \mathcal{U}_{N-1} \cdots \mathcal{U}_0,$$

where E_γ is the endpoint identification. If the same Hilbert space and basis are used at the end, $E_\gamma = \mathbf{I}$. Diagnostics are:

$$s_{\min} = \min_{j,k} (\Sigma_j)_{kk}, \quad \eta_\gamma = \min_\theta \|U_\gamma - e^{i\theta} \mathbf{I}\|,$$

$$C_{12} = U_{\gamma_1} U_{\gamma_2} U_{\gamma_1}^{-1} U_{\gamma_2}^{-1}, \quad \eta_{12} = \min_\theta \|C_{12} - e^{i\theta} \mathbf{I}\|.$$

The algorithm is implemented in the companion package under `src/phason_holonomy/berry.py`.

B D(S3) Sector Data

For a finite group G , simple objects of $D(G)$ are pairs (C, π) . The quantum dimension and topological spin are

$$d_{(C,\pi)} = |C| \dim \pi, \quad \theta_{(C,\pi)} = \frac{\chi_\pi(g)}{\dim \pi}, \quad g \in C.$$

For S_3 , these formulas give Table 1. The sum of squared quantum dimensions is

$$1^2 + 1^2 + 2^2 + 3^2 + 3^2 + 2^2 + 2^2 + 2^2 = 36,$$

so $\mathcal{D}_{\text{tot}} = 6$, as required for a quantum double of a group of order six. Candidate phason actions must preserve this data and the full fusion rules.

C Fibonacci Matrix Conventions

The Fibonacci F -matrix used in the figures is real, symmetric, and involutive:

$$F = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix}, \quad F^{-1} = F.$$

With $R = \text{diag}(e^{-4\pi i/5}, e^{3\pi i/5})$, the two elementary braid generators in the three- τ , total- τ space are

$$B_1 = R, \quad B_2 = FRF.$$

The projective comparison distance used by the code is

$$\delta(U, B) = \min_{\theta} \|U - e^{i\theta} B\|_F.$$

D Companion Implementation

The project contains a minimal reproducible implementation. The file `scripts/generate_figures.py` generates all figures. The module `src/phason_holonomy/tilings.py` generates Penrose and Ammann-Beenker approximants. The module `src/phason_holonomy/berry.py` implements discrete Wilczek-Zee transport. The module `src/phason_holonomy/anyon_data.py` records the $D(S_3)$ sector data and Fibonacci F - and R -symbols. The module `src/phason_holonomy/validation.py` produces benchmark noncommuting holonomies for testing plotting and matrix diagnostics.

These scripts are intentionally modular. A future exact-diagonalization backend should replace the benchmark frames by eigenvectors of $H(\lambda_j)$, while leaving the gauge-consistent transport and commutator analysis unchanged.

E Toric-Code Homology on Aperiodic Cellulations

This appendix records the stabilizer count used in the toric-code prototype. Let K be a finite cellulation of a closed connected oriented surface Σ_g . There are $|K_1|$ qubits. The star stabilizers satisfy one global relation

$$\prod_{v \in K_0} A_v = \text{I},$$

because each edge touches two vertices. The plaquette stabilizers also satisfy one global relation

$$\prod_{p \in K_2} B_p = \text{I},$$

because each edge appears twice in the oriented boundary of all faces. If the cellulation has no additional degeneracies, the number of independent stabilizers is

$$(|K_0| - 1) + (|K_2| - 1).$$

The number of encoded qubits is therefore

$$k = |K_1| - |K_0| - |K_2| + 2.$$

Using Euler characteristic $|K_0| - |K_1| + |K_2| = 2 - 2g$, one obtains

$$k = 2g, \quad \dim \mathcal{H}_0 = 2^{2g}.$$

No periodic unit cell enters this derivation. The only requirements are the incidence relations of a valid cellulation.

A local phason flip in a disk changes K by an elementary subdivision, collapse, or diagonal exchange. These moves induce chain-homotopy equivalent cellular complexes. Hence

$$H_1(K; \mathbb{Z}_2) \cong H_1(K'; \mathbb{Z}_2)$$

and the logical Wilson algebra is unchanged. If a branch cut exchanges primal and dual strings, however, the map is no longer an ordinary chain map of the same stabilizer code. It is a duality wall. This is why the same geometric-looking flip can be trivial in the defect-free sector and nontrivial at a phason twist.

F Gap and Discriminant Reporting Protocol

A numerical study should publish the following data for every loop γ :

1. The approximant family, including patch radius, boundary condition, number of vertices, number of edges, and number of faces.
2. The phason coordinates or flip sequence defining γ , including endpoint relabeling.
3. The geometric discriminant components avoided by the loop.
4. The minimum bulk gap $\min_{\gamma} \Delta_{\text{bulk}}$ and maximum internal splitting $\max_{\gamma} \delta_{\text{split}}$.
5. The smallest singular value of each frame-overlap matrix M_j .
6. The projective scalar-distance η_{γ} and commutator-distance η_{12} .
7. The action on Wilson operators, ribbon projectors, or fusion-tree observables.
8. The dependence on loop deformation, weak disorder, perturbation amplitude, and approximant size.

Without these data, a matrix U_{γ} is not interpretable. With these data, both positive and null results become valuable. A positive result identifies where topology enters. A null result identifies which phason cycles are invisible to intrinsic topological order.

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